

METASTABLE DENSITIES FOR CONTACT PROCESSES ON RANDOM GRAPHS

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ABSTRACT. We consider the contact process on the random graph chosen according to the power law model of Newman, Strogatz and Watts (2001, [6]). We follow the work of Chatterjee and Durrett (2009, [2]) who showed that for arbitrarily small infection parameter $\lambda > 0$, the limiting metastable density does not tend to zero as the graph size becomes large. We show three distinct regimes for this density depending on the tail of the degree law.

1. INTRODUCTION

In this article we examine a question left open by [2] on the contact process on random graphs.

The model postulated in [2] was a graph $G^n = (V^n, E^n)$ where V^n could be thought of as the set $\{1, 2, \dots, n\}$. The degrees of vertices were chosen as follows. One assumed a fixed probability p on \mathbb{N} and considered the distribution of n random variables with law p . Then, the degrees (d_1, \dots, d_n) were sampled independently with this distribution conditioned on the event $\{\sum_{i=1}^n d_i \equiv 0 \pmod{2}\}$. The probability p was taken satisfying

- (1) $p(0) = p(1) = p(2) = 0$,
- (2) $\lim_{x \rightarrow \infty} x^a p(x) = c \in (0, \infty)$.

for some $a > 1$.

Given a suitable realization $(d_1(\omega), d_2(\omega), \dots, d_n(\omega))$, the graph G^n was formed via the recipe of [6]: each vertex i was issued with d_i half edges and these half edges were matched up in a uniformly chosen manner. Of course, this random matching up may result in a graph with loops or multiple edges between two distinct vertices. However, as we will see, their presence is not relevant to the questions we shall address.

We then consider the contact process on this graph. The contact process on a graph $G = (V, E)$ is a continuous time Markov chain on $\{0, 1\}^V$. Assuming G has no loops or multiple edges, the process has operator

$$\Omega f(\xi) = \sum_{x \in V} (f(\xi^x) - f(\xi)) c_x(\xi)$$

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where $\forall x \in V$, $\xi^x \in \{0, 1\}^V$ is the element of $\{0, 1\}^V$ so that $\xi^x(y) = \xi(y)$ if and only if $y \neq x$ and

$$c_x(\xi) = \begin{cases} 1 & \text{if } \xi(x) = 1; \\ \lambda \cdot \sum_{y: (x,y) \in E} \xi(y) & \text{if } \xi(x) = 0. \end{cases}$$

The parameter λ is called the infection parameter. The terminology is due to the fact that the contact process is usually taken as a model for the spread of an infection in a population. In this interpretation, sites of the graph are thought of as individuals and states 0 and 1 are respectively interpreted as “healthy” and “infected”.

In case the graph has loops or multiple edges between pairs of sites, we need to specify in the definition of the dynamics how they are treated. We take the process to be as above with

$$c_x(\xi) = \begin{cases} 1 & \text{if } \xi(x) = 1; \\ \lambda \cdot \sum_{e \in E} \sum_{y: \{x,y\} \in e} \xi(y) & \text{if } \xi(x) = 0. \end{cases}$$

A configuration $\xi \in \{0, 1\}^V$ (or indeed a process (ξ_t)) may be identified with the subset of V (process of subsets) consisting of vertices for which $\xi(x) = 1$. The process as defined above has \emptyset as a trap state which for a finite graph G will eventually be reached. We may write this time as either

$$\tau = \inf\{t : \xi_t = \emptyset\} \quad \text{or} \quad \tau = \inf\{t : \xi_t = \emptyset\}.$$

Equally we may write $x \in \xi$ to signify that $\xi(x) = 1$.

It is elementary that the above operator defines a Markov process but it will be very instructive to introduce a construction of the process, the so-called Harris construction. We suppose given independent Poisson processes D_x , $x \in V$ of rate 1 and D_e , $e \in E$ of rate λ . The process $\{\xi_t : t \geq 0\}$ is constructed by stipulating that

- (i) at $t \in D_x$, $\xi_t(x)$ flips to 0 if immediately prior to t it had value 1 (if immediately prior $\xi_t(x)$ was 0 its value is unchanged),
- (ii) at $t \in D_e$ with $x, y \in e$ both $\xi_t(x)$ and $\xi_t(y)$ become 1 if one among them is a 1 and the other 0; otherwise they are unchanged.

Arrivals of the processes $(D_x)_{x \in V}$ are called *deaths*; arrivals of the processes $(D_e)_{e \in E}$ are called *transmissions*. We say that there is an *infection path* from (x, s) to (y, t) with $(x, s), (y, t) \in V \times \mathbb{R}_+$, $s < t$ if there exists n and sequences $s = s_0 < s_1 < s_2 \cdots < s_n < s_{n+1} = t$ and $x = x_0, x_1, \dots, x_n = y$ with the properties that

- (a) for each $1 \leq i \leq n$ there exists e_i so that $s_i \in D_{e_i}$ and $x_i, x_{i-1} \in e_i$,
- (b) for each $1 \leq i \leq n+1$, $[s_{i-1}, s_i] \cap D_{x_{i-1}} = \emptyset$.

We write $(x, s) \leftrightarrow (y, t)$ if there is an infection path from (x, s) to (y, t) . By extension we take (for $A, B \subset V \times \mathbb{R}_+$) $A \leftrightarrow B$ to mean that $(x, s) \leftrightarrow (y, t)$ for some $(x, s) \in A$ and $(y, t) \in B$. For $C \subset V$, we say $A \leftrightarrow B$ inside C if

there exists a connecting path entirely contained in C . It is easy to verify that for the contact process constructed via a Harris system of Poisson processes

$$\xi_t(y) = 1 \iff \exists x \in \xi_0 : (x, 0) \leftrightarrow (y, t).$$

This property may be reexpressed using the following notation. Given a Harris system, $(\xi_t^x)_{t \geq 0}$ denotes the contact process constructed from $\xi_0(z) = \delta_{xz}$. Given $A \subset V$, $(\xi_t^A)_{t \geq 0}$ that constructed from $\xi_0^A(z) = I_{z \in A}$:

$$\forall A \subset V, \xi_t^A = \cup_{x \in A} \xi_t^x.$$

This is the so-called additivity property. We also have (using the same notation) the attractiveness property:

$$A \subset B \subset V \implies \forall t, \xi_t^A \subset \xi_t^B.$$

In the following, given a graph G we will write $P_{G,\lambda}$ for the probability under which processes $(\xi_t^A)_{t \geq 0}$ are contact processes on G with infection parameter λ and initially occupied sites equal to A . Sometimes we will write $(\xi_t^G) = (\xi_t^V)$ for the contact process started from full occupancy.

We now go back to the discussion of the contact process on the Newman-Strogatz-Watts random graph. Though many results were shown in [2], for us the most important was the following. Given $\delta \in (0, 1)$, $\lambda > 0$, the contact process $(\xi_t^{V_n})$ with parameter λ starting from full occupancy on the randomly chosen graph and $x \in V^n$ randomly chosen, let

$$\rho_n(\lambda, \delta) = \mathbb{P} \left(\xi_{e^{n^{1-\delta}}}^{V_n}(x) = 1 \right).$$

Then, no matter how small λ might be chosen,

$$\underline{\rho}(\lambda, \delta) := \liminf_{n \rightarrow \infty} \rho_n(\lambda, \delta) > 0.$$

Also define

$$\overline{\rho}(\lambda, \delta) := \limsup_{n \rightarrow \infty} \rho_n(\lambda, \delta).$$

Bounds for these quantities were obtained in [2]:

$$\forall \epsilon > 0, \delta > 0, \lambda^{1+(a-2)-\epsilon} \geq \overline{\rho}(\lambda, \delta) \geq \underline{\rho}(\lambda, \delta) \geq \lambda^{1+2(a-2)+\epsilon}$$

when λ is small enough.

The question was raised as to the actual behaviour of $\underline{\rho}(\lambda, \delta)$, $\overline{\rho}(\lambda, \delta)$ and to the behaviour for $2 < a < 3$. We show

Theorem 1. *If $a > 3$, there are nontrivial constants m_1, M_1 so that, for any $\delta > 0$ and small enough $\lambda > 0$,*

$$m_1 \frac{\lambda^{1+2(a-2)}}{\log^{2(a-2)} \left(\frac{1}{\lambda} \right)} \leq \underline{\rho}(\lambda, \delta) \leq \overline{\rho}(\lambda, \delta) \leq M_1 \frac{\lambda^{1+2(a-2)}}{\log^{2(a-2)} \left(\frac{1}{\lambda} \right)}.$$

Theorem 2. *If $2\frac{1}{2} < a < 3$, there are nontrivial constants m_2, M_2 so that, for any $\delta > 0$ and small enough $\lambda > 0$,*

$$m_2 \frac{\lambda^{1+2(a-2)}}{\log^{a-2}(\frac{1}{\lambda})} \leq \underline{\rho}(\lambda, \delta) \leq \overline{\rho}(\lambda, \delta) \leq M_2 \frac{\lambda^{1+2(a-2)}}{\log^{a-2}(\frac{1}{\lambda})}.$$

Theorem 3. *If $2 < a \leq 2\frac{1}{2}$, there are nontrivial constants m_3, M_3 so that, for any $\delta > 0$ and small enough $\lambda > 0$,*

$$m_3 \lambda^{1+\frac{a-2}{3-a}} \leq \underline{\rho}(\lambda, \delta) \leq \overline{\rho}(\lambda, \delta) \leq M_3 \lambda^{1+\frac{a-2}{3-a}}.$$

Of critical importance in the analysis of [2] was the occurrence of stars: subgraphs where there existed a central site x of high degree and so that every other vertex had degree 1 and was a neighbour of x : if broadly speaking $\deg(x)\lambda^2 \gg 1$ the contact process on the star will survive a long time; otherwise it will quickly die. For $a > 2\frac{1}{2}$ the bulk of the contribution to $\rho_n(\lambda, \delta)$ comes from points adjacent to sites of degree $\gg \frac{1}{\lambda^2}$ (we will be more precise later); however, for $a < 2\frac{1}{2}$ stars do not play a central role in the analysis: it is rather a question of the contact process quickly spreading out.

We conclude with some basic notation. Given a graph G containing vertex x , $B_G(x, K)$ will denote the set of vertices in G at distance less than or equal to K from x , including x itself. We have fixed law $p(\cdot)$ satisfying (1) and (2). Throughout μ will signify the mean $\mu = \sum_x xp(x)$ and $q(\cdot)$ will denote (for $a > 2$) the law $q(x) = \frac{xp(x)}{\mu}$. Its mean (which exists for $a > 3$) will be written as ν .

A trivial and immediate consequence of the assumptions (1) and (2) made on the kernel $p(\cdot)$ is the following. There exist $c_0, C_0 > 0$ such that, for large enough n ,

$$(3) \quad c_0 n^{-a} < p(n) < C_0 n^{-a} \quad (\text{for any } a > 1);$$

$$(4) \quad c_0 n^{-(a-1)} < \sum_{m=n}^{\infty} p(m) < C_0 n^{-(a-1)} \quad (\text{if } a > 2);$$

$$(5) \quad c_0 n^{-(a-1)} < q(n) < C_0 n^{-(a-1)} \quad (\text{if } a > 2);$$

$$(6) \quad c_0 n^{-(a-2)} < \sum_{m=n}^{\infty} q(m) < C_0 n^{-(a-2)} \quad (\text{if } a > 3);$$

$$(7) \quad c_0 n^{3-a} < \frac{\sum_{m=0}^n mq(m)}{\sum_{m=0}^n q(m)} < C_0 n^{3-a} \quad (\text{if } a \in (2, 3)).$$

2. SOME BASIC TOOLS AND A REDUCTION

In this section we collect together some general considerations which will be relevant in all three of our analyses.

2.1. Locally G^n is a tree. We start stating a result that was of fundamental importance in the analysis of [2]. See [2] and Chapter 3 of [4] for details.

Proposition 4. *For each $n \in \mathbb{N}$, let G^n be a Newman-Strogatz-Watts random graph with degree law p and assume that x is uniformly chosen in $\{1, \dots, n\}$, independently of the graph. As $n \rightarrow \infty$, the law of $B_{G^n}(x, K)$ converges to the law of $B_T(o, K)$, where T is a Galton-Watson tree such that*

- (i) *the degree of the root o is chosen $\sim p$.*
- (ii) *the degrees of subsequent vertices are iid $\sim q$.*

Notation: We will denote by $\mathbb{P}_{n,x,\lambda}$ a probability measure under which, independently: (i) a Newman-Strogatz-Watts random graph G^n on n vertices is chosen, (ii) a vertex x is chosen uniformly among the n vertices, and (iii) a Harris system of parameter λ is defined on G^n . We can then consider, for instance, the contact process on G^n started from x infected.

We will denote by $\tilde{\mathbb{P}}_{(h,q)}$ the law under which a random rooted tree T is obtained by choosing the degrees of the vertices independently: that of the root o according to probability $h(\cdot)$ and those of “subsequent” vertices according to $q(\cdot)$. We use $\tilde{\mathbb{P}}_q$ for the law under which every vertex has degree i.i.d. according to q . $\tilde{\mathbb{P}}_{(h,q),\lambda}$ will denote the joint law of a rate λ contact process on a tree independently generated according to law $\tilde{\mathbb{P}}_{(h,q)}$ (typically, h will equal p). We similarly use notation $\tilde{\mathbb{P}}_{q,\lambda}$.

2.2. Time to extinction on star graphs. As already noted, a basic concept in our analysis is that of a *star*. A star is a connected graph so that all vertices except a privileged one (called the *hub*) have degree 1. The *degree* of the star will be the degree of the hub. One of the important contributions of [2] was the realization that, for the contact process on Newman-Strogatz-Watts graphs, the presence of subgraphs that are stars of large degree is of fundamental importance. As in that paper, for us it will be important to understand how long the contact process can survive on a hub of large degree. The following two lemmas give respectively a lower and an upper bound for the time of survival.

Lemma 5. *There exists a constant $C_{2.1}$ so that, for $\lambda < 1/100$, $M > 100$ and a star S of degree $\frac{M}{\lambda^2}$ and hub x ,*

$$P_{S,\lambda} \left(\int_0^{e^{MC_{2.1}}} I_{\{\xi_t^x(x)=1\}} dt \geq \frac{e^{MC_{2.1}}}{2} \right) \geq \frac{e^{-1}}{2}.$$

Here and in the rest of the paper, I denotes indicator function.

Proof. Consider the following events

$$A_1(x) = \{\text{there is no death at } x \text{ in interval } [0, 1]\};$$

$$A_{1a}(x) = \left\{ \begin{array}{l} \text{at least } \frac{M\lambda}{2}e^{-1} \text{ sites become infected} \\ \text{in } [0, 1] \text{ and have no death marks in } [0, 1] \end{array} \right\};$$

and, for $j > 1$,

$$A_j(x) = \left\{ x \text{ is in state 1 at least } \frac{90}{100} \text{ of the time in } [j-1, j] \right\};$$

$$A_{ja}(x) = \left\{ \begin{array}{l} \text{for at least } \frac{M\lambda}{2}e^{-1} \text{ sites } y, \text{ there is no death at } y \\ \text{in } [j-1, j] \text{ and there exists } t_y \in [j-1, j] \text{ such that} \\ \xi_{t_y}^x(x) = 1 \text{ and there is a transmission} \\ \text{from } x \text{ to } y \text{ at time } t_y \end{array} \right\}.$$

Obviously $P_{S,\lambda}(A_1(x)) = e^{-1}$ and for some universal $C_a > 0$,

$$P_{S,\lambda}(A_{1a}(x)) \geq 1 - e^{-C_a M}$$

uniformly over $\lambda \leq \frac{1}{100}$, $M \geq 100$. It is also easy to see that

$$\begin{aligned} & P_{S,\lambda}(A_j(x)^c \mid A_{j-1} a(x)) \\ & \leq P_{S,\lambda} \left(\begin{array}{l} \text{of at least } \frac{M\lambda}{2}e^{-1} \text{ occupied sites at time } j-1, \\ \text{less than } \frac{M\lambda e^{-2}}{4} \text{ survive in } [j-1, j] \end{array} \right) \\ & \quad + P_{S,\lambda} \left(\int_0^1 I_{\{z_s=1\}} ds \leq \frac{9}{10} \right), \end{aligned}$$

for $(z_s)_{s \geq 0}$ the Markov chain on $\{0, 1\}$ with rates $q_{01} = \frac{Me^{-2}}{4}$; $q_{10} = 1$. Thus,

$$P_{S,\lambda}(A_j(x) \mid A_{j-1} a(x)) \geq 1 - e^{-C_b M}$$

for some C_b uniformly over $\lambda \leq \frac{1}{100}$, $M \geq 100$.

Similarly,

$$P_{S,\lambda}(A_{ja}(x) \setminus A_j(x) \mid A_{j-1} a(x)) < e^{-C_d M}$$

for some universal C_d . The event $\left\{ \int_0^{e^{MC_{2.1}}} I_{\{\xi_t(x)=1\}} dt < \frac{e^{MC_{2.1}}}{2} \right\}$ is contained in $\cup_{j \leq MC_{2.1}} (A_j^c \cup A_{ja}^c)$ and so, fixing $C_{2.1}$ small, we obtain easily the desired result. \square

Lemma 6. *There exist $k > 0$, $C_{2.2} > 0$ so that, for $\lambda < 1/100$, $M \geq 1$ and a star S of degree $\leq \frac{M}{\lambda^2}$,*

$$P_{S,\lambda} \left(\xi_{3 \log(1/\lambda)}^S = \emptyset \right) \geq k e^{-C_{2.2} M}.$$

Proof. Obviously after time $2 \log \left(\frac{1}{\lambda} \right)$ the number of occupied leaves is stochastically less than a $\text{Bin} \left(\frac{M}{\lambda^2}, 2\lambda \right)$ and so will be less than $\frac{3M}{\lambda}$ with probability larger than $\frac{1}{3}$ uniformly over M, λ . Given this, the conditional probability that in next $\log \frac{1}{\lambda}$ units of time

- (a) x dies by time $2 \log \left(\frac{1}{\lambda} \right) + 1$ without infecting more than $\frac{M}{\lambda}$ leaves;
- (b) each leaf either infected at time $2 \log \left(\frac{1}{\lambda} \right)$ or that receives a transmission from 1 during time $[2 \log \left(\frac{1}{\lambda} \right), 2 \log \left(\frac{1}{\lambda} \right) + 1]$ dies during $[2 \log \left(\frac{1}{\lambda} \right) + 1, 3 \log \left(\frac{1}{\lambda} \right)]$ and makes no transmission to x during $[2 \log \left(\frac{1}{\lambda} \right), 3 \log \left(\frac{1}{\lambda} \right)]$

is greater than $(1 - e^{-\frac{1}{2}})C(1 - 2\lambda)^{\frac{4M}{\lambda}}$ for C universal (uniformly over relevant M and λ) and so the result follows. \square

2.3. Estimates for the contact process on trees of bounded degree.

We will need the following result, that concerns the contact process with small infection rate on a tree of bounded degree.

Lemma 7. *Let $\lambda > 0$ and T be a finite tree such that all sites have degree less than $\frac{1}{4\lambda^2}$. Then, if λ is small enough,*

$$\forall x, y \in T, P_{T,\lambda}((x, 0) \leftrightarrow \{y\} \times [0, \infty)) \leq (2\lambda)^{d(x,y)}.$$

Proof. Given the Harris construction the above probability is equal to that of

$$\exists t > 0 : \xi_t^x(y) = 1$$

for $\{\xi_t^x : t \geq 0\}$ the contact process on T of rate λ starting with just x occupied. We note that

$$M_t = \sum_{z \in T} \xi_{t \wedge \tau}^x(z) (2\lambda)^{d(z,y)}$$

is a supermartingale where $\tau = \inf\{t : \xi_t^x(y) = 1\}$. Thus the desired result follows from the optional sampling theorem. \square

We can adapt the above reasoning to get

Lemma 8. *Let $\lambda > 0$ and T be a finite tree such that all sites have degree less than $\frac{1}{8\lambda^2}$. Then, if λ is small enough, for all $t \geq 0$*

$$\forall x, y \in T, P_{T,\lambda}(\xi_t^x(y) = 1) \leq (2\lambda)^{d(x,y)} e^{-t/4},$$

where (ξ_t^x) is the rate λ contact process on the tree T with initially only the site x occupied.

This has the important corollary

Corollary 9. *Consider the contact process $(\xi_t^T)_{t \geq 0}$ on a tree T starting from full occupancy. Suppose that T satisfies the condition of Lemma 8 and that λ is small in the sense of this result. Then,*

$$P_{T,\lambda}(\xi_t^T \neq \emptyset) \leq |T|^2 e^{-t/4}.$$

Corollary 10. *Let T be a tree of maximum degree bounded by $\frac{1}{8\lambda^2}$. Then, if λ is small enough,*

$$(8) \quad P_{T,\lambda}((x, 0) \leftrightarrow \{y\} \times [t, \infty)) \leq 4(2\lambda)^{d(x,y)} e^{-t/4} \quad \forall t > 0, x, y \in T;$$

$$(9) \quad \begin{aligned} &P_{T,\lambda}(\{x\} \times [0, t] \leftrightarrow \{y\} \times [t', \infty)) \\ &\leq 8t(2\lambda)^{d(x,y)} e^{-\max(0, \frac{t'-t}{4})} \quad \forall t, t' > 0, x, y \in T. \end{aligned}$$

Additionally, for any $\delta > 0$, if λ is small enough we have

$$(10) \quad P_{T,\lambda} \left(\begin{array}{c} \exists t' : \{x\} \times [0, t] \leftrightarrow (y, t') \text{ and} \\ (y, t') \leftrightarrow \{x\} \times [t', \infty) \end{array} \right) \leq (2\lambda)^{(2-\delta)d(x,y)} \quad \forall t > 0, x, y \in T.$$

Proof. Let $\sigma_t^{x,y} = \inf\{s \geq t : \xi_s^x(y) = 1\}$ and $\psi_t^{x,y} = \inf\{s > \sigma_t^{x,y} : \xi_s^x(y) = 0\}$. We have

$$\begin{aligned} P_{T,\lambda}(\sigma_t^{x,y} < \infty) &= E_{T,\lambda} \left(I_{\{\sigma_t^{x,y} < \infty\}} \cdot E_{T,\lambda}(\psi_t^{x,y} - \sigma_t^{x,y} \mid \sigma_t^{x,y}) \right) \\ &\leq E_{T,\lambda} \left(\int_t^\infty I_{\{\xi_s^x(y)=1\}} ds \right) = \int_t^\infty P_{T,\lambda}(\xi_s^x(y) = 1) ds \\ &\leq (2\lambda)^{d(x,y)} \int_0^t e^{-s/4} ds = (2\lambda)^{d(x,y)} e^{-t/4}, \end{aligned}$$

proving (8). For (9), define A_x as the event that there is at least one transmission starting at x before the first death mark at x after time 0. We have

$$P_{T,\lambda} \left(\begin{array}{c} (x, 0) \leftrightarrow \\ \{y\} \times [t, \infty) \end{array} \mid A_x \right) = \frac{1 + \lambda \deg(x)}{\lambda \deg(x)} \cdot P_{T,\lambda}((x, 0) \leftrightarrow \{y\} \times [t, \infty)).$$

Given times $a < b$, recall that $N^x(b) - N^x(a)$ is the number of transmissions starting from x in $[a, b]$. Let $\tau(x, a)$ denote the time of the first transmission starting from x after time a . Then, for any $c > 0$,

$$\begin{aligned} P_{T,\lambda}(\{x\} \times [a, b] \leftrightarrow \{y\} \times [c, \infty)) &\leq P_{T,\lambda}(N^x(b) - N^x(a) \geq 2) + \\ &\int_a^b P_{T,\lambda} \left(\begin{array}{c} (x, 0) \leftrightarrow \\ \{y\} \times [\max(0, c-s), \infty) \end{array} \mid A_x \right) \cdot P_{T,\lambda} \left(\begin{array}{c} N^x(b) - N^x(a) = 1, \\ \tau(x, a) \in ds \end{array} \right) \\ &\leq C\lambda^2(b-a) + \\ &4t(2\lambda)^{d(x,y)} \frac{1 + \lambda \deg(x)}{\lambda \deg(x)} e^{-\frac{1}{4} \max(0, c-b)} P_{T,\lambda}(N^x(b) - N^x(a) = 1). \end{aligned}$$

Finally,

$$\begin{aligned} &P_{T,\lambda}(\{x\} \times [0, t] \leftrightarrow \{y\} \times [t', \infty)) \\ &\leq \limsup_{t \rightarrow \infty} \sum_{i=0}^N P_{T,\lambda} \left(\{x\} \times \left[i \frac{t}{N}, (i+1) \frac{t}{N} \right) \leftrightarrow \{y\} \times [t', \infty) \right) \\ &\leq 4t(2\lambda)^{d(x,y)} \frac{1 + \lambda \deg(x)}{\lambda \deg(x)} \int_0^t e^{-\frac{1}{4} \max(0, t'-s)} \lambda ds \\ &\leq 8t(2\lambda)^{d(x,y)} e^{-\max(0, \frac{t'-t}{4})}. \end{aligned}$$

We now prove (10). We have

$$\begin{aligned} & P_{T,\lambda}(\exists t' : (x, 0) \leftrightarrow (y, t') \leftrightarrow \{x\} \times [t', \infty)) \\ & \leq P_{T,\lambda}(\exists t' > 10d(x, y) \log(1/\lambda) : \xi_{t'}^x(y) = 1) \\ & + P_{T,\lambda}\left(\sigma_0^{x,y} < \infty, \right. \\ & \quad \left. \{y\} \times [\sigma_0^{x,y}, \sigma_0^{x,y} + 10d(x, y) \log(1/\lambda)] \leftrightarrow \{x\} \times [\sigma_0^{x,y}, \infty) \right) \end{aligned}$$

By (8), the first term in the sum is less than $4(2\lambda)^{d(x,y)} \cdot \lambda^{\frac{10}{4}d(x,y)} < \lambda^{2d(x,y)}$ when λ is small. The second term in the sum can be treated with a similar argument that proved (9); it is smaller than

$$(2\lambda)^{d(x,y)} \cdot 8(10 d(x, y) \log(1/\lambda)) \cdot (2\lambda)^{d(x,y)} < (2\lambda)^{(2-\delta)d(x,y)}$$

when λ is small. \square

2.4. Reduction to contact process on Galton-Watson trees. In [2], the authors show the following. Fix $\epsilon > 0$ and $\delta > 0$ and start the contact process on the Newman-Strogatz-Watts graph on n vertices, $G^n = (V^n, E^n)$, with a single occupied site of degree $\geq \frac{1}{\lambda^{2+\epsilon}}$. Then, provided λ is sufficiently small and n is sufficiently large, the process will survive with probability $\geq C(\epsilon)$ up until time $e^{n^{1-\delta}}$. It is not hard to adapt their approach to show

Lemma 11. *For $n \in \mathbb{N}$, $x \in V^n$, $\epsilon > 0$ and $K \in \mathbb{N}$, define*

$$A_{n,x,\epsilon,K} = \left\{ \begin{array}{l} \exists y \in B_{G^n}(x, K) : \deg(y) > 1/\lambda^{2+\epsilon}, \\ (x, 0) \leftrightarrow \{y\} \times \mathbb{R}_+ \text{ inside } B_{G^n}(x, K) \end{array} \right\}.$$

Then, given ϵ , δ , κ and $K \in \mathbb{N}$, there exist $\lambda_1 > 0$ and $n_0 \in \mathbb{N}$ so that

$$\rho_n(\lambda, \delta) \geq (1 - \kappa) \cdot \mathbb{P}_{n,x,\lambda}(A_{n,x,\epsilon,K})$$

when $n \geq n_0$ and $\lambda < \lambda_1$.

Together with Proposition 4, this easily generates the following result.

Proposition 12. *For any ϵ , $\delta > 0$, there exists $\lambda_1 > 0$ such that, for any $\lambda < \lambda_1$ and $R > 1$,*

$$\underline{\rho}(\lambda, \delta) \geq \tilde{\mathbb{P}}_{(p,q),\lambda} \left(\begin{array}{l} \exists y \in B_T(o, R) : \deg(y) > \frac{1}{\lambda^{2+\epsilon}}, \\ (o, 0) \leftrightarrow \{y\} \times \mathbb{R}_+ \text{ inside } B_T(o, R) \end{array} \right).$$

Again using Proposition 4, we easily get

Proposition 13. *For any $\delta > 0$*

$$\overline{\rho}(\lambda, \delta) \leq \tilde{\mathbb{P}}_{(p,q),\lambda}(\xi_t^o \neq \emptyset \forall t).$$

In the following three sections, we prove Theorems 1, 2 and 3 by applying the two above propositions. Thus, in this sections we only study the contact process on Galton-Watson trees and do not mention the Newman-Strogatz-Watts random graphs.

3. THE CASE $a > 3$, PROOF OF THEOREM 1

3.1. Lower bound. When $a > 3$, sites of “supercritical” degree, that is, degree larger than $1/\lambda$ to some power strictly larger than 2, are typically very far from each other and far from the root of the tree. However, as the next lemma shows, if the infection starts at a site whose degree is a large multiple of $\frac{1}{\lambda^2} \log^2(\frac{1}{\lambda})$, then the infection is maintained for a long time and can therefore reach distant sites. We then argue that at this distance, supercritical sites can be found.

Lemma 14. *For any $M > 100$, $\lambda > 0$ small enough and any connected graph $G = (V, E)$ with $x, y \in V$, $\deg(x) \geq 4M \frac{1}{\lambda^2} \log^2(\frac{1}{\lambda})$, $d(x, y) \leq M \log(\frac{1}{\lambda})$, we have*

$$P_{G,\lambda}(\exists t : \xi_t^x(y) = 1) \geq \frac{1}{10}.$$

Proof. Let $S = B(x, 1)$, $r = 2M \log(1/\lambda)$ and $I = \lfloor C_{2.1} \frac{(1/\lambda)^{4M \log(1/\lambda)}}{r} \rfloor$, where $C_{2.1}$ is the constant of Lemma 5. Define the events

$$\begin{aligned} A &= \{\forall s \leq Ir, \exists z \in S : \xi_s^x(z) = 1\}; \\ A_i &= \{\exists z \in S : \xi_{ir}^x(z) = 1\}, \quad i = 0, \dots, I-1, \end{aligned}$$

so that $A \subset \cap_{i=1}^I A_i$ and, by Lemma 5 and the fact that

$$C_1 \cdot \exp(\lambda^2 \cdot 4M(1/\lambda)^2 \log^2(1/\lambda)) \geq Ir,$$

we obtain, for $\lambda < 1/100$, $P_{G,\lambda}(A) \geq P_{S,\lambda}(A) \geq \frac{1}{9}$.

On A_i , we can choose $Z_i \in S$ such that $\xi_{ir}^x(Z_i) = 1$ and a path $\gamma_0 = Z_i, \gamma_1, \dots, \gamma_k = y$ such that $d(\gamma_j, \gamma_{j+1}) = 1 \forall j$ and $k \leq M \log(1/\lambda) + 1 \leq 2M \log(1/\lambda)$. Then, defining

$$B_i = A_i \cap \{\exists s \in [ir, (i+1)r) : (Z_i, ir) \leftrightarrow (y, s)\},$$

we have

$$P_{G,\lambda}(B_i) \geq (e^{-1}(1 - e^{-\lambda}))^{2M \log(1/\lambda)} \geq (\lambda/8)^{2M \log(1/\lambda)},$$

since an infection path can be obtained by imposing that, for $0 \leq j < k$, there is no death mark in $\{\gamma_j\} \times [ir+j, ir+j+1)$ and at least one transmission in $\{(\gamma_j, \gamma_{j+1})\} \times [ir+j, ir+j+1)$. Then,

$$\begin{aligned} P_{G,\lambda} \left(A \cap (\cup_{i=0}^{I-1} B_i)^c \right) &\leq P_{G,\lambda} \left((\cap_{i=0}^{I-1} A_i) \cap (\cup_{i=0}^{I-1} B_i)^c \right) \\ &\leq P_{G,\lambda} \left((\cap_{i=0}^{I-1} A_i) \cap (\cup_{i=0}^{I-2} B_i)^c \right) \cdot P_{G,\lambda}(B_{I-1}^c) \\ &\leq P_{G,\lambda} \left((\cap_{i=0}^{I-2} A_i) \cap (\cup_{i=0}^{I-2} B_i)^c \right) \cdot \left(1 - (\lambda/8)^{2M \log(1/\lambda)} \right) \end{aligned}$$

Iterating, we get

$$\begin{aligned} P_{G,\lambda} \left(A \cap \left(\bigcup_{i=0}^{I-1} B_i \right)^c \right) &\leq \left(1 - (\lambda/8)^{2M \log(1/\lambda)} \right)^I \\ &\leq \exp \left\{ - (\lambda/8)^{2M \log(1/\lambda)} \cdot \frac{C_{2,1}}{2} \cdot \frac{(1/\lambda)^{4M \log(1/\lambda)}}{2M \log(1/\lambda)} \right\} \end{aligned}$$

and this tends to zero as $\lambda \rightarrow 0$. In conclusion, when λ is small enough, $P_{G,\lambda}(\exists t : \xi_t^x(y) = 1) \geq P_{G,\lambda}(A \cap (\bigcup_{i=0}^{I-1} B_i) \geq 1/10)$. \square

Now, let $M = 100a$, $D = 4M(1/\lambda^2) \log^2(1/\lambda)$ and $R = M \log(1/\lambda)$. We have

$$\begin{aligned} \tilde{\mathbb{P}}_{(p,q),\lambda} \left(\begin{array}{l} \exists t > 0, y \in B_T(o, R) : \deg(y) > 1/\lambda^3, \\ (o, 0) \leftrightarrow (y, t) \text{ inside } B_T(o, R) \end{array} \right) &\geq \\ \tilde{\mathbb{E}}_{(p,q)} \left(I \left\{ \begin{array}{l} \exists x \in B_T(o, 1), y \in B_T(o, R) : \\ \deg(x) \geq D, \deg(y) > 1/\lambda^3 \end{array} \right\} \cdot P_{B_T(o,R),\lambda}(\exists t : \xi_t^o(x) = 1) \right. & \\ \left. \cdot P_{B_T(o,R),\lambda}(\exists t : \xi_t^x(y) = 1) \right). & \end{aligned}$$

On the event in the indicator function,

$$P_{B_T(o,R),\lambda}(\exists t : \xi_t^o(x) = 1) \geq \lambda/(1 + \lambda)$$

and also, by the above lemma,

$$P_{B_T(o,R),\lambda}(\exists t : \xi_t^x(y) = 1) \geq \frac{1}{10}$$

provided $\lambda < 1/100$. Also,

$$\begin{aligned} \tilde{\mathbb{P}}_{(p,q)} \left(\begin{array}{l} \exists x \in B_T(o, 1), y \in B_T(o, R) : \deg(x) \geq D, \deg(y) > 1/\lambda^3 \end{array} \right) & \\ \geq \tilde{\mathbb{P}}_{(p,q)} \left(\begin{array}{l} \exists x \in B_T(o, 1) : \\ \deg(x) \geq D \end{array} \right) - \tilde{\mathbb{P}}_{(p,q)} \left(\begin{array}{l} \nexists y \in B_T(o, R) : \\ \deg(y) > 1/\lambda^3 \end{array} \right) & \\ \geq c_0 D^{a-2} - (1 - c_0 \lambda^{3(a-2)})^{2^R}, & \end{aligned}$$

since there are at least 2^R vertices at distance R from o . The above is larger than

$$c_0 \left(4(100a) \frac{1}{\lambda^2} \log^2 \left(\frac{1}{\lambda} \right) \right)^{a-2} - \exp \left\{ -c_0 \lambda^{3(a-2)} \cdot 2^{100a \log \frac{1}{\lambda}} \right\} > m \frac{\lambda^{2(a-2)}}{\log^2(1/\lambda)}$$

for some $m > 0$ and λ small enough. We thus have, for some $m_0 > 0$ and small λ ,

$$\tilde{\mathbb{P}}_{(p,q),\lambda} \left(\begin{array}{l} \exists t > 0, y \in B_T(o, R) : \deg(y) > 1/\lambda^3, \\ (o, 0) \leftrightarrow (y, t) \text{ inside } B_T(o, R) \end{array} \right) \geq m_0 \frac{\lambda^{1+2(a-2)}}{\log^2(1/\lambda)},$$

so now the lower bound in Theorem 1 follows from Proposition 12.

3.2. Upper bound. We now want to argue in the opposite direction as that of the previous subsection. That is: sites of degree smaller than a small multiple of $\frac{1}{\lambda^2} \log^2 \left(\frac{1}{\lambda} \right)$ do not maintain the infection for a time that is sufficient to reach supercritical sites. As a first step in this direction, the next result bounds the time of survival on trees in which all sites except the root have small degree.

Lemma 15. *There exist $C_{3.1}, C'_{3.1} > 0$ such that, for small enough λ , if $\epsilon_0 > 0$, T is a tree with root o such that $\deg(o) = \epsilon_0 \cdot \frac{1}{\lambda^2} \cdot \log^2 \left(\frac{1}{\lambda} \right)$, $\deg(x) < \frac{1}{8\lambda^2}$ for all $x \neq o$ and $\#T \leq \frac{1}{\lambda^3}$, then*

$$P_{T,\lambda} \left(\xi_{(1/\lambda)C_{3.1}\epsilon_0 \cdot \log(1/\lambda)}^T \neq \emptyset \right) \leq \exp \left\{ - \left(\frac{1}{\lambda} \right)^{C'_{3.1}\epsilon_0 \cdot \log(1/\lambda)} \right\}.$$

Proof. We will divide time into integer multiples of $1/\lambda$; in each interval of the form $[3i/\lambda, 6i/\lambda]$ we will make an attempt to extinguish the infection in the whole tree. Each attempt will have small probability of success, so we will need a large number of them.

Let $\Gamma = \{x : d(o, x) = 1\}$, $\Psi = \{x : d(o, x) \geq 2\}$, $\gamma = \#\Gamma$, $\psi = \#\Psi$. For $i \in \mathbb{N}$, define

$$\sigma^i = \min(5i/\lambda, \inf\{t \geq 4i/\lambda : \text{there is a death mark at } o\}).$$

Let $G_1^i = \{\sigma^i < 5i/\lambda\}$, so that $\mathbb{P}(G_1^i) \geq 1 - e^{-1/\lambda}$. Define

$$X_x^i(t) = I_{\{T \times \{3i/\lambda\} \leftrightarrow (x,t) \text{ or } \{o\} \times [3i/\lambda, t] \leftrightarrow (x,t)\}}, \quad x \in T \setminus \{o\}, \quad t \in [4i/\lambda, 5i/\lambda].$$

By duality and Lemma 7, for $x \in T \setminus \{o\}$ and $t \in [4i/\lambda, 5i/\lambda]$,

$$\begin{aligned} P_{T,\lambda}(X_x^i(t) = 1) &= P_{T,\lambda} \left(\begin{array}{l} (x, 0) \leftrightarrow \{o\} \times [0, \infty) \text{ or} \\ (x, 0) \leftrightarrow T \times \{t - 3i/\lambda\} \text{ through a} \\ \text{path that does not pass by } o \end{array} \right) \\ &\leq (2\lambda)^{d(o,x)} + (1/\lambda^3)^2 e^{-(t-3i/\lambda)} \leq \begin{cases} 4\lambda & \text{if } x \in \Gamma; \\ 2(2\lambda)^2 & \text{if } x \in \Psi \end{cases} \end{aligned}$$

if λ is small. In particular, if λ is small,

$$(11) \quad E_{T,\lambda} \left(\sum_{x \in \Gamma} X_x^i(t) \right) \leq 4\lambda\gamma; \quad E_{T,\lambda} \left(\sum_{x \in \Psi} X_x^i(t) \right) \leq 8\lambda^2\psi.$$

From now on, we assume that λ is such that all these inequalities are satisfied. Next, define

$$G_2^i = G_1^i \cap \left\{ \sum_{x \in \Gamma} X_x^i(\sigma^i) < 16\lambda\gamma \right\}; \quad G_3^i = G_1^i \cap \left\{ \sum_{x \in \Psi} X_x^i(\sigma^i) < 32\lambda^2\psi \right\}.$$

For any $t \in [4i/\lambda, 5i/\lambda]$, we have

$$\begin{aligned} P_{T,\lambda}(G_2^i \cap G_3^i \mid \sigma^i = t) \\ = P_{T,\lambda}\left(\sum_{x \in \Gamma} X_x^i(t) < 16\lambda\gamma, \sum_{x \in \Psi} X_x^i(t) < 32\lambda^2\psi \mid \sigma^i = t\right); \end{aligned}$$

noticing that for any x , $X_x^i(t)$ is independent of the times of death marks at the root o , we can remove the conditioning from the right-hand side. Also using (11) and Markov's inequality, the above is larger than

$$\begin{aligned} 1 - P_{T,\lambda}\left(\sum_{x \in \Gamma} X_x^i(t) < 16\lambda\gamma\right) - P_{T,\lambda}\left(\sum_{x \in \Psi} X_x^i(t) < 32\lambda^2\psi\right) \\ > 1 - \frac{4\lambda\gamma}{16\lambda\gamma} - \frac{8\lambda^2\psi}{32\lambda^2\psi} = \frac{1}{2}. \end{aligned}$$

We thus get

$$P_{T,\lambda}(G_2^i \cap G_3^i) = \int_{4i/\lambda}^{5i/\lambda} P_{T,\lambda}(G_2^i \cap G_3^i \mid \sigma^i = t) P_{T,\lambda}(\sigma^i = dt) \geq P_{T,\lambda}(G_1^i)/2.$$

Now define

$$\begin{aligned} G_4^i &= G_2^i \cap \{\forall x \in \Gamma : X_x^i(\sigma^i) = 1, (x, \sigma^i) \leftrightarrow T \times \{6i/\lambda\}\}; \\ G_5^i &= G_3^i \cap \{\forall x \in \Psi : X_x^i(\sigma^i) = 1, (x, \sigma^i) \leftrightarrow T \times \{6i/\lambda\}\}. \end{aligned}$$

Notice that the events $\{(x, \sigma^i) \leftrightarrow T \times \{6i/\lambda\}\}$ for $x \in T \setminus \{o\}$ are increasing with respect to the death marks in the Harris system and decreasing with respect to the transmissions. They are thus positively correlated. For this reason, given $t \in [4i/\lambda, 5i/\lambda]$, $A \subset \Gamma$ with $\#A < 16\lambda\gamma$ and $B \subset \Psi$ with $\#B < 32\lambda\psi$, we have

$$\begin{aligned} P_{T,\lambda}\left(G_4^i \cap G_5^i \mid \begin{array}{l} G_2^i \cap G_3^i \cap \{\sigma^i = t\} \\ \cap \{\{x \in \Gamma : X_x^i(x) = 1\} = A\} \\ \cap \{\{x \in \Psi : X_x^i(x) = 1\} = B\} \end{array}\right) \\ \geq \prod_{x \in A \cup B} P_{T,\lambda}((x, t) \leftrightarrow T \times \{6i/\lambda\}) \\ \geq \prod_{x \in A \cup B} \left(1 - P_{T,\lambda}\left(\begin{array}{l} (x, t) \leftrightarrow \{o\} \times [t, \infty) \text{ or} \\ (x, t) \leftrightarrow T \times \{6i/\lambda\} \text{ through a path} \\ \text{that does not pass by } o \end{array}\right)\right) \\ \geq (1 - 4\lambda)^{16\lambda\gamma} \cdot (1 - 8\lambda^2)^{32\lambda^2\psi}. \end{aligned}$$

Using the inequality $1 - \ell < e^{-2\ell}$, which holds for small enough $\ell > 0$, this is larger than

$$\begin{aligned} &\exp\{-2 \cdot 4 \cdot 16 \cdot \lambda^2 \cdot \gamma - 2 \cdot 8 \cdot 32 \cdot \lambda^4 \cdot \psi\} \\ &\geq \exp\left\{-C \left(\lambda^2 \cdot \epsilon_0 \cdot \frac{1}{\lambda^2} \cdot \log^2\left(\frac{1}{\lambda}\right) + \lambda^4 \cdot \frac{1}{\lambda^3}\right)\right\} \geq \lambda^{\bar{C} \cdot \epsilon_0 \cdot \log(1/\lambda)} \end{aligned}$$

for some universal $\bar{C} > 0$. Now put $C_{3.1} = 8\bar{C}$ and $C'_{3.1} = \bar{C}$.

Define $G^i = \cap_{j=1}^5 G_j^i$. The events $(G^i)_{i=0}^\infty$ are independent and, from what we have seen, it follows that

$$(12) \quad P_{T,\lambda}(G^i) \geq (1 - e^{-1/\lambda}) \cdot (1/2) \cdot \lambda^{\frac{C_{3.1}}{8} \cdot \epsilon_0 \cdot \log(1/\lambda)} > (1/4) \lambda^{\frac{C_{3.1}}{8} \cdot \epsilon_0 \cdot \log(1/\lambda)}.$$

Finally, if G^i occurs, then $T \times \{3i/\lambda\} \leftrightarrow T \times \{6i/\lambda\}$. This implies that, if $\left\{ \xi_{(1/\lambda)C_{3.1}\epsilon_0 \cdot \log(1/\lambda)}^T \neq \emptyset \right\}$ occurs, then G^i cannot occur for

$$0 \leq i \leq \left\lfloor \frac{(1/\lambda)^{C_{3.1}\epsilon_0 \cdot \log(1/\lambda)}}{3/\lambda} - 1 \right\rfloor = \left\lfloor \frac{1}{3} \cdot \left(\frac{1}{\lambda} \right)^{C_{3.1}\epsilon_0 \cdot \log(1/\lambda) - 1} - 1 \right\rfloor.$$

This has probability smaller than

$$\begin{aligned} & \left(1 - \frac{1}{4} \lambda^{\frac{C_{3.1}}{8} \cdot \epsilon_0 \cdot \log(1/\lambda)} \right) \left\lfloor \frac{1}{3} \cdot \left(\frac{1}{\lambda} \right)^{C_{3.1}\epsilon_0 \cdot \log(1/\lambda) - 1} - 1 \right\rfloor \\ & \leq \exp \left\{ -\frac{1}{4} \cdot \lambda^{\frac{C_{3.1}}{8} \cdot \epsilon_0 \cdot \log(1/\lambda)} \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \left(\frac{1}{\lambda} \right)^{\frac{C_{3.1}}{2} \cdot \epsilon_0 \cdot \log(1/\lambda)} \right\} \\ & \leq \exp \left\{ -\left(\frac{1}{\lambda} \right)^{C'_{3.1} \epsilon_0 \cdot \log(1/\lambda)} \right\} \end{aligned}$$

when λ is small. □

Recall that μ and ν denote the expectations associated to laws p and q , respectively. Denote by $\theta = \max(\mu, \nu)$. Since $a > 3$, we can choose $\epsilon \in (0, 1/2)$ such that

$$(13) \quad 4(a-2) - 3\epsilon \log \theta > 2(a-2) + 1;$$

$$(14) \quad 3(a-2) - 4\epsilon(2a-3) \log \theta > 2(a-2) + 1.$$

We also choose ϵ_0 so that

$$(15) \quad \epsilon_0 < \epsilon/C_{3.1}.$$

Define the events on Galton-Watson trees:

$$A_1 = \{ \deg(o) > 1/(8\lambda^2) \};$$

$$A_2 = \{ \#B(o, 2\epsilon \log(1/\lambda)) > 1/\lambda^3 \};$$

$$A_3 = \left\{ \begin{array}{l} \text{There exist } x, y \in B(o, 2\epsilon \log(1/\lambda)) : \\ \deg(x), \deg(y) > 1/(8\lambda^2) \text{ and the geodesics} \\ \text{that connect } o \text{ to } x \text{ and } o \text{ to } y \text{ only intersect at } o \end{array} \right\};$$

$$A = A_1 \cup A_2 \cup A_3.$$

We will show that the probability of A is negligible in comparison to that of the event that contributes the most to the probability of survival (namely, the root infecting a neighbor of large degree). This gives a precise meaning to our earlier suggestion that sites of large degree are typically isolated and far from the root.

$$B_1 = \{\forall x \in B(o, 1), \deg(x) \leq 1/(8\lambda^2)\};$$

$$B_2 = \{\text{There exists a unique } o^* : d(o, o^*) = 1, \deg(o^*) > 1/(8\lambda^2)\}.$$

The whole space is contained in $A \cup B_1 \cup B_2$. We further define the following sub-events of B_2 :

$$B_{21} = B_2 \cap \{\deg(o^*) > \epsilon_0(1/\lambda^2) \log^2(1/\lambda)\};$$

$$B_{22} = B_2 \cap \left\{ \begin{array}{l} \exists y \in B(o, 2\epsilon \log(1/\lambda)) : \deg(y) > 1/(8\lambda^2), \\ \text{the geodesic that connects } o \text{ to } y \text{ contains } o^* \end{array} \right\};$$

$$B_{23} = B_2 \setminus (B_{21} \cup B_{22}).$$

We then have

$$(16) \quad \begin{aligned} \tilde{\mathbb{P}}_{(p,q),\lambda} \left(\xi_t^{\{o\}} \neq \emptyset \forall t \right) &\leq \tilde{\mathbb{P}}_{(p,q)}(A_1) + \tilde{\mathbb{P}}_{(p,q)}(A_2) + \tilde{\mathbb{P}}_{(p,q)}(A_3) \\ &\quad + \tilde{\mathbb{P}}_{(p,q),\lambda} (\xi_t^o \neq \emptyset \forall t \mid B_1 \cap A_1^c) \\ &\quad + \tilde{\mathbb{P}}_{(p,q)}(B_{21}) \cdot \tilde{\mathbb{P}}_{(p,q),\lambda} (\xi_t^o \neq \emptyset \forall t \mid B_{21} \cap A^c) \\ &\quad + \tilde{\mathbb{P}}_{(p,q)}(B_{22} \cap B_{21}^c) \cdot \tilde{\mathbb{P}}_{(p,q),\lambda} (\xi_t^o \neq \emptyset \forall t \mid B_{22} \cap A^c) \\ &\quad + \tilde{\mathbb{P}}_{(p,q),\lambda} \left(\xi_t^{\{o,o^*\}} \neq \emptyset \forall t \mid B_{23} \cap A^c \right). \end{aligned}$$

We will now show that the right-hand side of the above equation is less than $C \frac{\lambda^{1+2(a-2)}}{\log^{2(a-2)}(1/\lambda)}$. We will treat each of the terms in turn.

§1) $\tilde{\mathbb{P}}_{(p,q)}(A_1)$. Since $\deg(o)$ has law p under $\tilde{\mathbb{P}}_{(p,q)}$ we have, for λ small enough,

$$(17) \quad \tilde{\mathbb{P}}_{(p,q)}(A_1) \leq p[1/(8\lambda^2), \infty) \leq C\lambda^{2(a-1)} \leq \frac{\lambda^{1+2(a-2)}}{\log^{2(a-2)}(1/\lambda)}.$$

§2) $\tilde{\mathbb{P}}_{(p,q)}(A_2)$. We will need the following

Lemma 16. *Let Z be distributed as q and let Z_1, Z_2, \dots be a sequence of independent random variables distributed as $Z - \nu$. Then, there exists $C_{3.2} > 0$ such that, for all $n \in \mathbb{N}$ and $x > 0$,*

$$\mathbb{P}(Z_1 + \dots + Z_n > x) \leq C_{3.2} \frac{n}{x^{a-2}}.$$

Proof. Let ϕ_{Z_1} be the characteristic function of Z_1 . By (3.2.2) in [3], there exists $C > 0$ such that, for all t, x ,

$$(18) \quad |1 + itx - e^{itx}| \leq C \min(tx, (tx)^2).$$

Choose C'_0 such that $\mathbb{P}(|Z_1| = x) \leq C'_0/|x|^{a-1}$ for any $x \in \mathbb{R}$. Applying (18), we have

$$\begin{aligned}
 (19) \quad |1 - \phi_{Z_1}(t)| &= \left| \sum_{n \in \nu + \mathbb{Z}} (1 + itn - e^{itn}) \mathbb{P}(Z_1 = n) \right| \\
 &\leq \sum_{n \in \nu + \mathbb{Z}} |1 + itn - e^{itn}| \mathbb{P}(Z_1 = n) \\
 &\leq C \cdot C'_0 \cdot \sum_{\substack{n \in \nu + \mathbb{Z}, \\ |n| \leq 1/t}} (tn)^2 \cdot \frac{1}{n^{a-1}} + C \cdot C'_0 \cdot \sum_{\substack{n \in \nu + \mathbb{Z}, \\ |n| > 1/t}} tn \cdot \frac{1}{n^{a-1}} \\
 (20) \quad &\leq C' \left(t^2 \frac{1}{t^{4-a}} + t \frac{1}{t^{3-a}} \right) \leq C' t^{a-2}.
 \end{aligned}$$

Now, fix $n \in \mathbb{N}$, write $S_n = Z_1 + \dots + Z_n$ and let ϕ_{S_n} be the characteristic function of S_n . Using the fact that, for σ in a neighborhood of 1, $|1 - \sigma^n| \leq 2n|1 - \sigma|$, we have, if $|t|$ is small enough,

$$|1 - \phi_{S_n}(t)| = |1 - \phi_{Z_1}(t)^n| \leq 2n|1 - \phi_{Z_1}(t)| \leq C' n t^{a-2}.$$

Then, by (2.2.1) in [3],

$$\mathbb{P}(S_n > x) \leq \frac{x}{2} \int_{-2/x}^{2/x} (1 - \phi_{S_n}(t)) dt \leq \frac{x}{2} \frac{4}{x} C' n \left(\frac{2}{x} \right)^{a-2} \leq C' \frac{n}{x^{a-2}}$$

as required. \square

Now, for the sake of readability, we put

$$\begin{aligned}
 x &= \frac{1}{2\lambda^3}, \quad g = \lfloor 2\epsilon \log(1/\lambda) \rfloor, \quad V = \mu^2 \sum_{i=1}^g \nu^{2(i-1)}, \\
 s_i &= \frac{\mu^2 \nu^{2(i-1)}}{V}, \quad X_i = \#B(o, i), \quad 1 \leq i \leq g.
 \end{aligned}$$

Then, we have

$$\begin{aligned}
 &\{\#B(o, 2\epsilon \log(1/\lambda)) > 1/\lambda^3\} \subset \{X_1 + \dots + X_g > x\} \\
 &\subset \{X_1 > s_1 x\} \cup \left(\bigcup_{i=2}^g \left\{ \begin{array}{l} X_i > s_i x, \\ X_{i-1} \leq s_{i-1} x \end{array} \right\} \right),
 \end{aligned}$$

so that

$$\begin{aligned}
 (21) \quad \tilde{\mathbb{P}}_{(p,q)}(A_2) &\leq \tilde{\mathbb{P}}_{(p,q)}(X_1 > s_1 x) \\
 &+ \sum_{i=2}^g \sum_{k=1}^{s_{i-1} x} \tilde{\mathbb{P}}_{(p,q)}(X_i > s_i x \mid X_{i-1} = k) \cdot \tilde{\mathbb{P}}_{(p,q)}(X_{i-1} = k).
 \end{aligned}$$

We have

$$\tilde{\mathbb{P}}_{p,q}(X_1 > s_1 x) \leq \tilde{\mathbb{P}}\left(X_1 > \frac{\mu^2}{V} x\right) \leq \tilde{\mathbb{P}}\left(X_1 > \frac{x}{V}\right) \leq C_0 \frac{V^{a-2}}{x^{a-2}};$$

in the second inequality we have used the fact that $\mu > 1$. Also, using the notation of Lemma 16, if $i \geq 2$,

$$\mathbb{P}_q(X_i > s_i x \mid X_{i-1} = k) = \mathbb{P}(Z_1 + \dots + Z_k > s_i x - \nu k).$$

When $k \leq s_{i-1}x$, we have $s_i x - \nu k \geq (x/V)\mu^2(\nu^{2(i-1)} - \nu^{2(i-2)}) \geq x/V$. So, applying Lemma 16, when $k \leq s_{i-1}x$ we have

$$\mathbb{P}(Z_1 + \dots + Z_k > s_i x - \nu k) \leq C_{3.2} \frac{kV^{a-2}}{x^{a-2}}.$$

The expression in (21) is thus less than

$$\begin{aligned} & C_0 \frac{V^{a-2}}{x^{a-2}} + C_{3.2} \frac{V^{a-2}}{x^{a-2}} \sum_{i=2}^g \sum_{k=1}^{s_{i-1}x} k \tilde{\mathbb{P}}(X_{i-1} = k) \\ & \leq C_0 \frac{V^{a-2}}{x^{a-2}} + C_{3.2} \frac{V^{a-2}}{x^{a-2}} \sum_{i=2}^g \mu \nu^{i-1} \leq C \cdot \frac{1}{x^{a-2}} \cdot V^{a-2} \cdot \theta^{g+1}. \end{aligned}$$

Now, note that $V \leq \theta^{2g+2}$, so the right hand side is less than

$$C \frac{\theta^{2(g+1)(a-2)+g+1}}{x^{a-2}} = C \lambda^{3(a-2)} \cdot \theta^{(g+1)(2a-3)} \leq C \lambda^{3(a-2)-4\epsilon(2a-3)\log(\theta)}.$$

Therefore, by (14), when λ is small enough,

$$(22) \quad \tilde{\mathbb{P}}_{(p,q)}(A_2) \leq \frac{\lambda^{1+2(a-2)}}{\log^{2(a-2)}(1/\lambda)}.$$

§3) $\tilde{\mathbb{P}}_{(p,q)}(A_3)$. This is bounded as follows.

$$\begin{aligned} \tilde{\mathbb{P}}_{(p,q)}(A_3) & \leq \sum_{n=2}^{\infty} \tilde{\mathbb{P}}_{(p,q)}(\deg(o) = n) \\ & \quad \times \tilde{\mathbb{P}}_{(p,q)} \left(\begin{array}{l} \exists x, y, z, w \in B(o, 2\epsilon \log(1/\lambda)) : \\ x \neq y, d(x, o) = d(y, o) = 1, \\ \text{the geodesics from } x \text{ to } z \text{ and} \\ \text{from } y \text{ to } w \text{ do not contain } o; \\ \deg(z), \deg(w) > 1/(8\lambda^2) \end{array} \middle| \deg(o) = n \right) \\ & \leq \sum_{n=2}^{\infty} p(n) \cdot \binom{n}{2} \cdot \tilde{\mathbb{P}}_q(\exists z \in B(o, 2\epsilon \log(1/\lambda) - 1) : \deg(z) > 1/(8\lambda^2))^2 \\ & \leq C_0 \lambda^{2 \cdot 2(a-2)} \cdot \left(1 + \theta + \dots + \theta^{2\epsilon \log(1/\lambda)}\right)^2 \cdot \sum_{n=1}^{\infty} n^2 p(n). \end{aligned}$$

Since $a > 3$, the sum in the last expression is finite; then, using (13), we conclude that, for small enough λ ,

$$(23) \quad \tilde{\mathbb{P}}_{(p,q)}(A_3) \leq C \lambda^{4(a-2)} \frac{1}{\lambda^{3\epsilon \log \theta}} \leq \frac{\lambda^{1+2(a-2)}}{\log^{2(a-2)}(1/\lambda)}.$$

§4) $\tilde{\mathbb{P}}_{(p,q),\lambda}(\xi_t^o \neq \emptyset \forall t \mid B_1 \cap A_1^c)$. Choose $k \in \mathbb{N}$ with $k > 2(a-2) + 1$. Fix $\lambda > 0$ and let T be a rooted tree such that the root and its neighbors have degree smaller than $1/(8\lambda^2)$. Define, for $2 \leq i \leq k-1$,

$$\mathcal{D}_i(T) = \left\{ \begin{array}{l} x \in T : d(o, x) = i; \deg(x) > 1/(8\lambda^2); \\ \text{the geodesic from } o \text{ to } x \text{ does not contain} \\ \text{any vertex of degree larger than } 1/(8\lambda^2) \end{array} \right\},$$

$$\mathcal{D}_k(T) = \{x \in T : d(o, x) = k\};$$

put $d_i = d_i(T) = \#\mathcal{D}_i(T)$ for $2 \leq i \leq k$. Finally, put

$$T' = (\cup_{i=2}^k \mathcal{D}_i) \cup \left\{ \begin{array}{l} x \in T : d(o, x) \leq k, \text{ the geodesic} \\ \text{from } o \text{ to } x \text{ does not intersect } (\cup_{i=2}^k \mathcal{D}_i) \end{array} \right\}.$$

Now, in T' the maximum degree is smaller than $1/(8\lambda^2)$, so Lemma 7 applies and we have $P_{T',\lambda}(\xi_t^o(x) = 1 \text{ for some } t) \leq (2\lambda)^{d(o,x)}$. Then,

$$P_{T,\lambda}(\xi_t^o \neq \emptyset \forall t) \leq P_{T',\lambda}(\exists t > 0, x \in \cup_{i=2}^k \mathcal{D}_i : \xi_t^o(x) = 1) \leq \sum_{i=2}^k d_i (2\lambda)^i,$$

so

$$(24) \quad \tilde{\mathbb{P}}_{(p,q),\lambda}(\xi_t^{\{o\}} \neq \emptyset \forall t \mid B_1 \cap A_1^c) \leq \sum_{i=2}^k (2\lambda)^i \cdot \tilde{\mathbb{E}}_{(p,q)}(d_i \mid B_1 \cap A_1^c).$$

Now, noticing that, for $2 \leq i \leq k-1$,

$$\begin{aligned} \tilde{\mathbb{E}}_{(p,q)}(d_i \mid B_1 \cap A_1^c) &\leq \tilde{\mathbb{E}}_{(p,q)}(\#\{x : d(o, x) = i, \deg(x) > 1/(8\lambda^2)\} \mid B_1 \cap A_1^c) \\ &\leq \tilde{\mathbb{E}}_{(p,q)}(\#\{x : d(o, x) = i, \deg(x) > 1/(8\lambda^2)\}) \\ &= \mu\nu^{i-1}q[1/(8\lambda^2), \infty) \leq C_0 \theta^i (8\lambda^2)^{a-2} \end{aligned}$$

and

$$\tilde{\mathbb{E}}_{(p,q)}(d_k \mid B_1 \cap A_1^c) \leq \tilde{\mathbb{E}}_{(p,q)}(d_k) \leq \theta^k,$$

(24) is less than

$$(2\theta\lambda)^k + C_0 \cdot 8^{a-2} \cdot \lambda^{2(a-2)} \cdot (2\theta\lambda)^2 \cdot \sum_{i=0}^{\infty} (2\theta\lambda)^i.$$

By the choice of k , we conclude that, for λ small enough,

$$(25) \quad \tilde{\mathbb{P}}_{(p,q),\lambda}(\xi_t^{\{o\}} \neq \emptyset \forall t \mid B_1 \cap A_1^c) \leq \frac{\lambda^{1+2(a-2)}}{\log^{2(a-2)}(1/\lambda)}.$$

§5) $\tilde{\mathbb{P}}_{(p,q)}(B_{21})$. We bound directly:

$$\begin{aligned}
 \tilde{\mathbb{P}}_{(p,q)}(B_{21}) &\leq \sum_{k=1}^{\infty} \tilde{\mathbb{P}}_{(p,q)}(\deg(o) = k) \\
 &\quad \cdot \tilde{\mathbb{P}}_{(p,q)} \left(\begin{array}{c} \exists o^* : d(o, o^*) = 1, \\ \deg(o^*) \geq \epsilon_0(1/\lambda^2) \log^2(1/\lambda) \end{array} \middle| \deg(o) = k \right) \\
 &\leq C_0 \frac{\lambda^{2(a-2)}}{\epsilon_0^{a-2} \cdot \log^{2(a-2)}(1/\lambda)} \sum_{k=1}^{\infty} k \tilde{\mathbb{P}}_{(p,q)}(\deg(o) = k) \\
 (26) \quad &\leq C \frac{\lambda^{2(a-2)}}{\log^{2(a-2)}(1/\lambda)}
 \end{aligned}$$

for some $C > 0$.

§6) $\tilde{\mathbb{P}}_{(p,q)}(B_{22} \cap B_{21}^c)$. For $B_{22} \cap B_{21}^c$ to occur, the root o must have a neighbour o^* of degree between $1/(8\lambda^2)$ and $\epsilon_0(1/\lambda^2) \log^2(1/\lambda)$, and there must exist a vertex y of degree larger than $1/(8\lambda^2)$ in the tree that grows from o^* away from o ; y must be at distance less than $2\epsilon \log(1/\lambda) - 1$ from o^* . We thus have

$$\begin{aligned}
 \tilde{\mathbb{P}}_{(p,q)}(B_{22} \cap B_{21}^c) &\leq \sum_{k=\lfloor 1/(8\lambda^2) \rfloor}^{\lceil \epsilon_0(1/\lambda^2) \log^2(1/\lambda) \rceil} \tilde{\mathbb{P}}_{(p,q)}(B_2, \deg(o^*) = k) \\
 &\quad \cdot \tilde{\mathbb{P}}_{(\delta_k, q)} \left(\begin{array}{c} \exists y \in B(o, 2\epsilon \log(1/\lambda)) : \\ \deg(y) > 1/(8\lambda^2) \end{array} \right) \\
 &\leq \tilde{\mathbb{P}}_{(\delta_{\epsilon_0(1/\lambda^2) \log^2(1/\lambda)}, q)} \left(\begin{array}{c} \exists y \in B(o, 2\epsilon \log(1/\lambda)) : \\ \deg(y) > 1/(8\lambda^2) \end{array} \right) \cdot \tilde{\mathbb{P}}_{(p,q)}(B_1^c) \\
 &\leq \frac{\epsilon_0}{\lambda^2} \cdot \log^2 \left(\frac{1}{\lambda} \right) \cdot \left(1 + \nu + \dots + \nu^{2\epsilon \log(1/\lambda)} \right) \cdot \lambda^{2(a-2)} \cdot \mu \cdot \lambda^{2(a-2)}.
 \end{aligned}$$

When λ is small enough, this last expression is less than $\lambda^{4(a-2)-2-3\epsilon \log \nu}$, so, by (13),

$$(27) \quad \tilde{\mathbb{P}}_{(p,q)}(B_{22} \cap B_{21}^c) \leq \frac{\lambda^{2(a-2)}}{\log^{2(a-2)}(1/\lambda)}.$$

§7) $\tilde{\mathbb{P}}_{(p,q),\lambda}(\xi_t^o \neq \emptyset \forall t \mid B_{21} \cap A^c)$, $\tilde{\mathbb{P}}_{(p,q),\lambda}(\xi_t^o \neq \emptyset \forall t \mid B_{22} \cap A^c)$. These two probabilities are bounded in the same way, so we will only treat the first. Let

$$T'' = \{o^*\} \cup \left\{ x \in B(o, 5) : \begin{array}{c} \text{the geodesic that connects} \\ o \text{ to } x \text{ does not contain } o^* \end{array} \right\}.$$

We have

$$\begin{aligned} \tilde{\mathbb{P}}_{(p,q),\lambda}(\xi_t^o \neq \emptyset \forall t \mid B_{21} \cap A^c) &\leq \tilde{\mathbb{P}}_{(p,q),\lambda} \left(\begin{array}{c} (o,0) \leftrightarrow \{o^*\} \times \mathbb{R}_+ \\ \text{inside } T'' \end{array} \mid B_{21} \cap A^c \right) \\ &\quad + \tilde{\mathbb{P}}_{(p,q),\lambda} \left(\begin{array}{c} \exists y \in T'' : d(o,y) = 5, \\ (o,0) \times \mathbb{R}_+ \text{ inside } T'' \end{array} \mid B_{21} \cap A^c \right). \end{aligned}$$

On $B_{21} \cap A^c$, T'' has maximal degree less than $1/(8\lambda^2)$ and has less than $1/\lambda^3$ vertices. Therefore, by Lemma 7, the right hand side is bounded by $2\lambda + (1/\lambda)^3 \cdot (2\lambda)^5 \leq 3\lambda$ when λ is small, and then

$$(28) \quad \tilde{\mathbb{P}}_{(p,q)}(\xi_t^o \neq \emptyset \forall t \mid B_{21} \cap A^c), \tilde{\mathbb{P}}_{(p,q),\lambda}(\xi_t^o \neq \emptyset \forall t \mid B_{22} \cap A^c) \leq 3\lambda.$$

§8) $\tilde{\mathbb{P}}_{(p,q),\lambda}(\xi_t^{\{o,o^*\}} \neq \emptyset \forall t \mid B_{23} \cap A^c)$. Let

$$r = \lfloor \epsilon \log(1/\lambda) \rfloor, \quad U = (1/\lambda)^{C_{3.1} \epsilon_0 \log(1/\lambda)},$$

where $C_{3.1}$ is as in Lemma 15. Define the following sub-events of $B_{23} \cap A^c$:

$$E_1 = B_{23} \cap A^c \cap \left\{ \begin{array}{c} \exists t > 0, x : d(o,x) = r : \text{There is an infection path} \\ \text{from } (o,0) \text{ to } (x,t) \text{ that does not pass by } o^* \end{array} \right\};$$

$$E_2 = B_{23} \cap A^c \cap \{ \{o, o^*\} \times \{0\} \leftrightarrow B_T(o, r) \times \{U\} \text{ inside } B_T(o, r) \};$$

$$E_3 = B_{23} \cap A^c \cap \{ \exists s \leq U, t > s, x : d(o^*, x) = r : (o^*, s) \leftrightarrow (x, t) \}.$$

We claim that

$$(29) \quad \left\{ \xi_U^{\{o,o^*\}} \neq \emptyset \right\} \subset E_1 \cup E_2 \cup E_3.$$

Indeed, assume that neither of E_1, E_2, E_3 occurs. Then, by the definition of E_1 and E_3 , no infection appears in $[0, U] \times B_T(o, r)^c$; in particular, $\xi_U^{\{o,o^*\}}(y) = 0$ for all $y \notin B_T(o, r)$. Also, if $y \in B_T(o, r)$, any infection path that reached (y, U) would need to stay inside of $B_T(o, r)$, which is impossible by the definition of E_2 . This completes the proof of (29).

Now assume that $B_{23} \cap A^c$ occurs and let T be a realization of the Galton-Watson tree. We know that $\deg(o^*) \leq \epsilon_0(1/\lambda^2) \log^2(1/\lambda)$, no other vertex in $B_T(o, 2r)$ has degree larger than $1/(8\lambda^2)$ and $\#B_T(o, 2r) \leq 1/\lambda^3$.

Applying Lemma 7 to the subtree obtained by removing from T all vertices that are reached from o through o^* (including o^* itself), and recalling the definition of A_2 , we get

$$P_{T,\lambda}(E_1) \leq \frac{1}{\lambda^3} \cdot (2\lambda)^{\lfloor \epsilon \log(1/\lambda) \rfloor}.$$

Next, applying Lemma 15 to T with the root placed at o^* , we have

$$P_{T,\lambda}(E_2) \leq \exp \left\{ - \left(\frac{1}{\lambda} \right)^{C'_{3.1} \epsilon_0 \log(1/\lambda)} \right\}.$$

Finally, for E_3 to occur, there must be an infection path that starts at (o^*, s) for some $s \in [0, U]$, does not return to o^* and ends at some point at distance

r from o^* . The restriction of not returning to o^* implies that this path stays inside a tree of maximum degree less than $1/(8\lambda^2)$, so we can use Corollary 10 to get

$$P_{T,\lambda}(E_3) \leq 8U \left(\frac{1}{\lambda}\right)^3 (2\lambda)^r \leq 8 \left(\frac{1}{\lambda}\right)^3 \lambda^{(\epsilon - C_{3.1} \cdot \epsilon_0) \log(1/\lambda)}.$$

Using all these bounds, we conclude that, for λ small enough,

$$(30) \quad \mathbb{P}_{(p,q),\lambda} \left(\xi_t^{\{o,o^*\}} \neq \emptyset \forall t \mid B_{23} \cap A^c \right) \leq \frac{\lambda^{1+2(a-2)}}{\log^{2(a-2)}(1/\lambda)}.$$

To conclude, using (17), (22), (23), (25), (26), (27) and (30) back in (16), we get

$$\mathbb{P}_{(p,q),\lambda} (\xi_t^o \neq \emptyset \forall t) \leq C \frac{\lambda^{1+2(a-2)}}{\log^{2(a-2)}(1/\lambda)}$$

for λ small enough.

4. THE CASE $2\frac{1}{2} < a < 3$, PROOF OF THEOREM 2

We note that in the present case (and in fact also when $a = 3$), the expectation associated to the size-biased distribution $q(\cdot)$ is infinite.

As was the case with the previous section, proving the lower bound is comparatively cheaper.

4.1. Lower bound. Fix $\delta > 0$ such that $\frac{2}{a-2} - \delta > 2$ and $M > 100$ such that $C_{2.1} M > 1$, where $C_{2.1}$ is the constant of Lemma 5. Define the following events on Galton-Watson trees and the contact process defined on these trees:

$$\begin{aligned} A_1 &= \left\{ \exists x_1 : d(o, x_1) = 1, \deg(x_1) \geq \frac{M}{\lambda^2} \log(1/\lambda) \right\}; \\ A_2 &= A_1 \cap \left\{ \exists x_2 \neq o : d(x_1, x_2) = 1, \deg(x_2) \geq \frac{1}{\lambda^{\frac{2}{a-2}-\delta}} \right\}; \\ A_3 &= A_1 \cap \left\{ \begin{array}{l} \text{A transmission from } o \text{ to } x_1 \text{ occurs at a} \\ \text{time } \tau \text{ earlier than the first death mark at } o \end{array} \right\}. \end{aligned}$$

On A_3 , define $(\zeta_t)_{t \geq \tau}$ as the contact process on $B(x_1, 1)$ started at time τ with only x_1 infected and built only with the restriction of the Harris system to $B(x_1, 1)$. Further define

$$\begin{aligned} A_4 &= A_3 \cap \left\{ \zeta_t(x_1) = 1 \text{ for half the time in } [\tau, \tau + e^{MC_{2.1} \log(1/\lambda)}] \right\}; \\ A_5 &= A_3 \cap \{ \zeta_t(x) = 1 \text{ for some } t \geq \tau \}. \end{aligned}$$

Obviously, if $\cap_{i=1}^5$ occurs, then $(o, 0) \leftrightarrow \{x_2\} \times \mathbb{R}_+$ inside $B(o, 2)$. We have

$$\begin{aligned}\tilde{\mathbb{P}}_{(p,q)}(A_1) &\geq c_0 \cdot \frac{\lambda^{2(a-2)}}{M^{a-2} \log^{a-2}(1/\lambda)}; \\ \tilde{\mathbb{P}}_{(p,q)}(A_2 \mid A_1) &\geq 1 - \left(1 - c_0 \lambda^{2-\delta(a-2)}\right)^{\frac{M}{\lambda^2}} \geq 1 - e^{-c_0 M (1/\lambda)^{\delta(a-2)}}; \\ \tilde{\mathbb{P}}_{(p,q),\lambda}(A_3 \mid A_1) &\geq \frac{\lambda}{1+\lambda}.\end{aligned}$$

Applying Lemma 5 (assuming that $\lambda > 1/100$),

$$\begin{aligned}\tilde{\mathbb{P}}_{(p,q),\lambda}(A_4 \mid A_3) &\geq 1/2e; \\ \tilde{\mathbb{P}}_{(p,q),\lambda}(A_5 \mid A_4) &\geq e^{-\frac{\lambda}{2}} \left(\frac{1}{\lambda}\right)^{MC_{2,1}}.\end{aligned}$$

Putting all this together gives

$$\tilde{\mathbb{P}}_{(p,q),\lambda} \left(\begin{array}{l} (o, 0) \text{ infects a site of degree} \\ > 1/\lambda^{\frac{2}{a-2}-\delta} \text{ inside } B(o, 2) \end{array} \right) \geq c \frac{\lambda^{1+2(a-2)}}{\log^{a-2}(1/\lambda)}$$

when λ is small, for some constant c that does not depend on λ . Our lower bound now follows from Proposition 12.

4.2. Upper bound. In our proof of the lower bound, a significant difference can be noted with respect to the case $a > 3$. It lies in the fact that the “supercritical” site that becomes infected is now close to the root. Since large sites are no longer far from the root, if the infection is sustained close to the root for some time of order $1/\lambda^\epsilon$, where ϵ is not very small, these sites will be reached. The proof of the upper bound then depends on controlling the probability of this event.

Let $\lambda > 0$, $R \in \mathbb{N}$ and T be a tree with root o . We define $T'_{R,\lambda}$ as the tree with vertices

$$T'_{R,\lambda} = \{o\} \cup \left\{ \begin{array}{l} x \in B_T(o, R) : \text{the geodesic from } o \text{ to } x \\ \text{does not contain any vertex of degree} \\ \text{larger than } \frac{1}{8\lambda^2} \text{ (except possibly } x \text{ itself)} \end{array} \right\}$$

and take the set of edges of $T'_{R,\lambda}$ to be the set of edges of T with both extremities lying in the above set of vertices. We also write

$$\begin{aligned}\deg'(x) &= \#\{y \in T'_{R,\lambda} : d(x, y) = 1\}, \\ \partial T'_{R,\lambda} &= \{x \in T'_{R,\lambda} : \deg'(x) = 1\}.\end{aligned}$$

We remark that, by Corollary 7 and Corollary 9, if $\deg(o) < 1/(8\lambda^2)$, then

$$(31) \quad P_{T,\lambda}((x, 0) \leftrightarrow \{y\} \times \mathbb{R}_+ \text{ inside } T'_{R,\lambda}) \leq (2\lambda)^{d(x,y)} \quad \forall x, y \in T'_{R,\lambda} \text{ and}$$

$$(32) \quad P_{T,\lambda}((o, 0) \leftrightarrow T'_{R,\lambda} \times [t, \infty) \text{ inside } T'_{R,\lambda}) \leq \#T'_{R,\lambda} \cdot e^{-t/4}.$$

We will often omit the dependence on R, λ and write T' and $\partial T'$.

For randomly chosen trees with deterministic root degree, we have

Lemma 17. *There exist $R \in \mathbb{N}$, $\epsilon, \epsilon' > 0$ such that, if λ is small enough and $D = \epsilon(1/\lambda^2) \log(1/\lambda)$,*

$$\tilde{\mathbb{P}}_{(\delta_D, q), \lambda} \left((o, 0) \leftrightarrow \partial T'_{R, \lambda} \times \mathbb{R}_+ \right) < \lambda^{\epsilon'}.$$

Proof. We take $\epsilon_0 = \frac{1-2(3-a)}{4}$ and R large enough that $R(1-2(3-a)) + 4 - 2a > 1 - 2(3-a)$. Also take $\epsilon = \frac{\epsilon_0}{4\bar{C}}$, where \bar{C} is the constant of Lemma 6.

Now, let T be a rooted tree. Given $y \in T' = T'_{R, \lambda}$, $y \neq o$, we define $\pi(y)$ as the neighbor of o contained in the geodesic from o to y , and $T'(y)$ as the subtree of T' with set of vertices $\{o\} \cup \{z : \text{the geodesic from } o \text{ to } z \text{ contains } \pi(y)\}$ and all edges with both extremities in this set of vertices. We remark that $T'(y)$ is a tree with maximum degree less than $1/(4\lambda^2)$. For any $t > 0$, by Corollary 10,

$$(33) \quad P_{T', \lambda}(\{o\} \times [0, t] \leftrightarrow \{y\} \times \mathbb{R}_+ \text{ inside } T'(y)) \leq 8t(2\lambda)^{d(o, y)};$$

$$(34) \quad P_{T, \lambda} \left(\begin{array}{c} \exists s \leq t, s'' > s' > s : \\ (o, s) \leftrightarrow (y, s') \leftrightarrow (o, s'') \text{ inside } T'(y) \end{array} \right) \leq (2\lambda)^{\frac{3}{2}d(o, y)}.$$

Given a graphical construction for the contact process on T , define (ζ_t) as the contact process on $B_T(o, 1)$ started with only the root infected and built only with the death marks and arrows of the graphical construction that are contained in $B_T(o, 1)$.

Finally, define

$$\phi_R(T) = \sum_{i=1}^R \lambda^i \cdot \#\{y \in \partial T' : d(o, y) = i\}.$$

We now consider the following events in a probability space where we have defined both a Galton-Watson tree T (with $\delta_{\epsilon(1/\lambda^2) \log(1/\lambda)}$ individuals in the first generation and offspring distribution q for subsequent generations) and the graphical construction for a contact process with parameter $\lambda > 0$.

$$E_1 = \{\phi_R(T) \geq \lambda^{\epsilon_0}\};$$

$$E_2 = \left\{ \zeta_{(1/\lambda)^{2\bar{C}\epsilon}} \neq \emptyset \right\};$$

$$E_3 = \left\{ \exists y \in \partial T' : \{o\} \times \left[0, (1/\lambda)^{2\bar{C}\epsilon}\right] \leftrightarrow \{y\} \times \mathbb{R}_+ \text{ inside } T'(y) \right\};$$

$$E_4 = \left\{ \begin{array}{c} \exists y \in T', s \leq (1/\lambda)^{2\bar{C}\epsilon}, s'' > s' > s : \\ (o, s) \leftrightarrow (y, s') \leftrightarrow (o, s'') \text{ inside } T'(y) \end{array} \right\}.$$

Claim 18. *If neither of E_1, E_2, E_3, E_4 occur, then $(o, 0) \leftrightarrow \partial T' \times \mathbb{R}_+$.*

Proof. Assume that none of the events occur. In order to have $(o, 0) \leftrightarrow \partial T' \times \mathbb{R}_+$, there needs to exist $y \in \partial T'$ and $t > 0$ such that $(o, 0) \leftrightarrow (y, t)$ inside T' ; let us show that this is impossible. Assume by contradiction that we can find such y and t and let γ be the corresponding infection path, so that $\gamma : [0, t] \rightarrow T$ with $\gamma(0) = o$ and $\gamma(t) = y$. Let $s^* = \sup\{s : \gamma(s) = o\}$. We cannot have $s^* \leq (1/\lambda)^{2\bar{C}\epsilon}$ because E_3 does not occur. So assume

that $s^* > (1/\lambda)^{2\bar{C}\epsilon}$. Now notice that $\gamma(s) \in B_T(o, 1) \forall s \leq (1/\lambda)^{2\bar{C}\epsilon}$ is impossible because E_2 does not occur, so there must exist $s^{**} \leq (1/\lambda)^{2\bar{C}\epsilon}$ such that $\gamma(s^{**}) \notin B_T(o, 1)$. Now, letting $\alpha = \sup\{s < s^{**} : \gamma(s) = o\}$ and $\beta = \inf\{s > s^{**} : \gamma(s) = o\}$, we have $(o, \alpha) \leftrightarrow (\gamma(s^{**}), s^{**}) \leftrightarrow (o, \beta)$, a contradiction because we assumed E_4 does not occur. \square

The probability in the statement of the lemma is thus less than

$$\tilde{\mathbb{P}}_{(\delta_D, q)}(E_1) + \tilde{\mathbb{P}}_{(\delta_D, q), \lambda}(E_2) + \tilde{\mathbb{P}}_{(\delta_D, q), \lambda}(E_3 \cap E_1^c) + \tilde{\mathbb{P}}_{(\delta_D, q), \lambda}(E_4)$$

Let us bound these terms.

§1) $\tilde{\mathbb{P}}_{(\delta_D, q)}(E_1)$. We bound

$$\tilde{\mathbb{E}}_{(\delta_D, q)}(\phi_R(T)) \leq \sum_{i=1}^R \lambda^i \cdot \tilde{\mathbb{E}}_{(\delta_D, q)}(\#\{y \in \partial T' : d(o, y) = i\}).$$

Now, under $\tilde{\mathbb{P}}_{(\delta_D, q)}$, T' has the distribution of a Galton-Watson tree such that: 1) the first generation has D individuals and 2) for subsequent generations, the offspring distribution is equal to the law q truncated at $1/(8\lambda^2)$. By (7), the expectation associated to this law is smaller than $C_0(1/(8\lambda^2))^{3-a}$. Thus, the right hand side of the above expression is bounded by

$$\begin{aligned} & C \sum_{i=1}^{R-1} \lambda^i \cdot \epsilon \cdot \frac{1}{\lambda^2} \cdot \log\left(\frac{1}{\lambda}\right) \cdot \left(\frac{1}{8\lambda^2}\right)^{(3-a)(i-1)} (8\lambda^2)^{a-2} \\ & \quad + C \cdot \lambda^R \cdot \epsilon \cdot \frac{1}{\lambda^2} \cdot \log\left(\frac{1}{\lambda}\right) \cdot \left(\frac{1}{8\lambda^2}\right)^{(3-a)(R-1)} \\ & \leq C \log(1/\lambda) \left\{ \sum_{i=1}^{R-1} \lambda^{i+2(a-2)-2(3-a)(i-1)-2} + \lambda^{R-2(3-a)(R-1)-2} \right\} \\ & \leq C \log(1/\lambda) \left(\lambda^{1-2(3-a)} + \lambda^{R(1-2(3-a))+4-2a} \right) \\ & \leq C \log(1/\lambda) \cdot \lambda^{1-2(3-a)} \leq \lambda^{\frac{1-2(3-a)}{2}} \end{aligned}$$

when λ is small. Therefore, we have

$$\tilde{\mathbb{P}}_{(\delta_D, q)}(\phi_R(T) \geq \lambda^{\epsilon_0}) \leq \frac{\tilde{\mathbb{E}}_{(\delta_D, q)}(\phi_R(T))}{\lambda^{\epsilon_0}} \leq \lambda^{\frac{1-2(3-a)}{2} - \frac{1-2(3-a)}{4}} = \lambda^{\frac{1-2(3-a)}{4}}.$$

§2) $\tilde{\mathbb{P}}_{(\delta_D, q), \lambda}(E_2)$. If E_2 occurs, then

$$\forall i \in \left\{ 0, 1, \dots, \left\lfloor \frac{(1/\lambda)^{2\bar{C}\epsilon}}{3 \log(1/\lambda)} \right\rfloor \right\}, \quad (\zeta_t) \text{ does not become extinct in } [i \cdot 3 \log(1/\lambda), (i+1) \cdot 3 \log(1/\lambda)].$$

Using $\frac{(1/\lambda)^{2\bar{C}\epsilon}}{4\log(1/\lambda)} < \left\lfloor \frac{(1/\lambda)^{2\bar{C}\epsilon}}{3\log(1/\lambda)} \right\rfloor$ and Lemma 6, the probability of this is less than

$$\begin{aligned} & \left(1 - ke^{-\bar{C}\epsilon \log(1/\lambda)}\right)^{\frac{(1/\lambda)^{2\bar{C}\epsilon}}{4\log(1/\lambda)}} = \left(1 - k\lambda^{\bar{C}\epsilon}\right)^{\frac{(1/\lambda)^{2\bar{C}\epsilon}}{4\log(1/\lambda)}} \\ & \leq \exp\left(-k\lambda^{\bar{C}\epsilon} \cdot \frac{(1/\lambda)^{2\bar{C}\epsilon}}{4\log(1/\lambda)}\right) < \lambda \end{aligned}$$

when λ is small enough.

§3) $\tilde{\mathbb{P}}_{(\delta_D, q), \lambda}(E_3 \cap E_1^c)$. Assume that E_1 does not occur and fix a realization of T . For $y \in \partial T'$, using (33) we have

$$P_{T', \lambda}(\{o\} \times [0, (1/\lambda)^{2\bar{C}\epsilon}] \leftrightarrow \{y\} \times \mathbb{R}_+ \text{ inside } T'(y)) \leq \left(\frac{1}{\lambda}\right)^{2\bar{C}\epsilon} (2\lambda)^{d(o, y)}$$

so that

$$\begin{aligned} & P_{T', \lambda} \left(\begin{array}{c} \exists y \in \partial T' : \{o\} \times [0, (1/\lambda)^{2\bar{C}\epsilon}] \leftrightarrow \{y\} \times \mathbb{R}_+ \\ \text{inside } T'(y) \end{array} \right) \\ & \leq C \left(\frac{1}{\lambda}\right)^{2\bar{C}\epsilon} \sum_{i=1}^R \lambda^i \cdot \#\{y \in \partial T' : d(o, y) = i\} \\ & = C \left(\frac{1}{\lambda}\right)^{2\bar{C}\epsilon} \phi_R(T) \leq C \left(\frac{1}{\lambda}\right)^{2\bar{C}\epsilon} \cdot \lambda^{\epsilon_0} = C\lambda^{\epsilon_0/2}. \end{aligned}$$

Finally,

$$\begin{aligned} & \tilde{\mathbb{E}}_{(\delta_D, q), \lambda}(E_3 \cap E_1^c) \\ & \leq \tilde{\mathbb{E}}_{(\delta_D, q), \lambda} \left(I_{E_1^c} \cdot P_{T', \lambda} \left(\begin{array}{c} \exists y \in \partial T' : \\ \{o\} \times [0, (1/\lambda)^{2\bar{C}\epsilon}] \leftrightarrow \{y\} \times \mathbb{R}_+ \\ \text{inside } T'(y) \end{array} \right) \right) \leq C\lambda^{\epsilon_0/2}. \end{aligned}$$

§4) $\tilde{\mathbb{P}}_{(\delta_D, q), \lambda}(E_4)$. Let T be a realization of the Galton-Watson tree. By (34),

$$\begin{aligned} & P_{T', \lambda} \left(\begin{array}{c} \exists y \in T' : d(o, y) \geq 2, \ s \leq (1/\lambda)^{2\bar{C}\epsilon}, \ s'' > s' > s : \\ (o, s) \leftrightarrow (y, s') \leftrightarrow (o, s'') \text{ inside } T'(y) \end{array} \right) \\ & \leq \left(\frac{1}{\lambda}\right)^{2\bar{C}\epsilon} \sum_{i=2}^R (2\lambda)^{\frac{3}{2}i} \cdot \#\{y \in T' : d(o, y) = i\}. \end{aligned}$$

Integrating over T ,

$$\begin{aligned} \tilde{\mathbb{P}}_{(\delta_D, q), \lambda}(E_4) &\leq \left(\frac{1}{\lambda}\right)^{2\bar{C}\epsilon} \sum_{i=2}^R (2\lambda)^{\frac{3}{2}i} \cdot \tilde{\mathbb{E}}_{(\delta_D, q)}(\#\{y \in T' : d(o, y) = i\}) \\ &\leq C \left(\frac{1}{\lambda}\right)^{2\bar{C}\epsilon} \sum_{i=2}^R \lambda^{\frac{3}{2}i} \cdot \left(\frac{1}{\lambda^2}\right)^{(3-a)(i-1)} \leq C \lambda^{2a-3-2\bar{C}\epsilon} = C \lambda^{2a-3-\epsilon_0/2}; \end{aligned}$$

recall that $\epsilon_0 = \frac{1-2(3-a)}{4} < 1/4$, so the above is smaller than $\lambda^{\epsilon'}$ for some $\epsilon' > 0$ and λ small. \square

In the rest of this section, we take R and ϵ as in the above lemma. Since in the proof of the lemma we could have taken R as large as desired, we may also assume that

$$(35) \quad R(1 - 2(3 - a)) > 2(a - 2) + 1.$$

We consider the following events in a probability space where we have defined both a Galton-Watson tree T (with offspring distribution p for the first generation and q for subsequent generations) and the graphical construction for a contact process with parameter $\lambda > 0$.

$$\begin{aligned} A_1 &= \{\deg(o) > 1/(8\lambda^2)\}; \\ A_2 &= A_1^c \cap \{ (o, 0) \leftrightarrow (B_T(o, 1)^c \cap \partial T') \times \mathbb{R}_+ \text{ inside } T' \}; \\ A_3 &= A_1^c \cap \{ (o, 0) \leftrightarrow T' \times [\log^2(1/\lambda), \infty) \text{ inside } T' \}; \\ A_4 &= A_1^c \cap \{ \#(B_T(o, 1) \cap \partial T') \geq 2 \} \cap \left\{ (o, 0) \leftrightarrow (B_T(o, 1) \cap \partial T') \times \mathbb{R}_+ \text{ inside } T' \right\}; \\ A_5 &= A_1^c \cap \{ \#(B_T(o, 1) \cap \partial T') = 1 \}. \end{aligned}$$

On A_5 , define x^* as the unique neighbor of o that is in $\partial T'$; note that $\deg'(x^*) = 1$ and $\deg(x^*) > 1/(8\lambda^2)$. Also on A_5 , define

$$\begin{aligned} \tau_1 &= \inf\{t : (o, 0) \leftrightarrow (x^*, t) \text{ inside } T'\} \in (0, \infty], \\ \tau_2 &= \inf\{t > \tau_1 : (o, 0) \leftrightarrow (x^*, t) \text{ inside } T'\} \in (0, \infty] \end{aligned}$$

and let T'' be the tree with root x^* , vertex set

$$\{x^*, o\} \cup \left\{ x \in B_T(o, R) : \begin{array}{l} \text{the geodesic from } x^* \text{ to } x \\ \text{does not contain } o \text{ nor any vertex of degree} \\ \text{larger than } \frac{1}{8\lambda^2} \text{ (except } x^* \text{ and possibly } x) \end{array} \right\};$$

and again take as edges of T'' the edges of T such that both extremities lie in the above set of vertices. Similarly define \deg'' and $\partial T''$ as before (note that $o \in \partial T''$).

Now define

$$\begin{aligned}
A_{51} &= A_5 \cap \left\{ \begin{array}{l} \text{there are at least 3 arrows from } o \text{ to } x^* \\ \text{in the time interval } [0, \log^2(1/\lambda)] \end{array} \right\}; \\
A_{52} &= A_5 \cap \{ \deg(x^*) > \epsilon(1/\lambda^2) \log(1/\lambda) \} \cap \{ \tau_1 < \infty \}; \\
A_{53} &= A_5 \cap \{ 1/(8\lambda^2) < \deg(x^*) \leq \epsilon(1/\lambda^2) \log(1/\lambda) \} \\
&\quad \cap \{ \tau_1 < \infty \} \cap \{ (x^*, \tau_1) \leftrightarrow \partial T'' \times [\tau_1, \infty) \text{ inside } T'' \}; \\
A_{54} &= A_5 \cap \{ 1/(8\lambda^2) < \deg(x^*) \leq \epsilon(1/\lambda^2) \log(1/\lambda) \} \\
&\quad \cap \{ \tau_2 < \infty \} \cap \{ (x^*, \tau_2) \leftrightarrow \partial T'' \times [\tau_2, \infty) \text{ inside } T'' \}.
\end{aligned}$$

Claim 19. *If neither of $A_1, A_2, A_3, A_4, A_{51}, A_{52}, A_{53}, A_{54}$ occurs, then $\xi_t^o \subset T' \cup T''$ for all $t > \log^2(1/\lambda)$, and thus the contact process dies out.*

Proof. Assume that neither of the events occur and define $\sigma = \inf\{t : (o, 0) \leftrightarrow \partial T'\}$. If $\sigma < \infty$, then there must exist $y \in \partial T'$ such that $(o, 0) \leftrightarrow (y, \sigma)$ inside T' . We consider several cases.

- Case 1: $\#(B_T(o, 1) \cap \partial T') = 0$. Since A_2 does not occur, we cannot have $d(o, y) \geq 2$, so y cannot exist, so the infection never leaves T' .
- Case 2: $\#(B_T(o, 1) \cap \partial T') = 2$. Since A_2 does not occur, we cannot have $d(o, y) \geq 2$. Since A_4 does not occur, we cannot have $d(o, y) = 1$ either, so y cannot exist and the infection never leaves T' .
- Case 3: $\#(B_T(o, 1) \cap \partial T') = 1$, $\deg(x^*) > \epsilon(1/\lambda^2) \log(1/\lambda)$. Again we cannot have $d(o, y) \geq 2$, and since A_{52} does not occur, we cannot have $y = x^*$. So again the infection does not leave T' .
- Case 4: $\#(B_T(o, 1) \cap \partial T') = 1$, $\deg(x^*) \leq \epsilon(1/\lambda^2) \log(1/\lambda)$. Assume that for some $t > \log^2(1/\lambda)$ and $z \in T$, we have $\xi_t^o(z) = 1$. Let $\gamma : [0, t] \rightarrow T$ be the corresponding infection path, so that $\gamma(0) = o$ and $\gamma(t) = z$. Let $\sigma' = \inf\{s : \gamma(s) \in \partial T'\}$. If we had $\sigma' > \log^2(1/\lambda)$, we would get $(o, 0) \leftrightarrow T' \times [\log^2(1/\lambda), \infty)$ inside T' , which is impossible because A_3 does not occur. Therefore, $\sigma' \leq \log^2(1/\lambda)$ and, arguing as in the former cases, we must have $\gamma(\sigma') = x^*$. Since A_{51} does not occur, we must either have $\sigma' = \tau_1$ or $\sigma' = \tau_2$. We then either use the definition of A_{53} or that of A_{54} to conclude that z must be in T'' . This completes the proof.

□

We now proceed to show that the probability of each of the events is less than a fixed multiple of $\frac{\lambda^{1+2(a-2)}}{\log^{(a-2)}(1/\lambda)}$.

§1) $\tilde{\mathbb{P}}_{(p,q)}(A_1)$. We have already shown in Subsection 3.2 that this is smaller than $\frac{\lambda^{1+2(a-2)}}{\log^{2(a-2)}(1/\lambda)} \leq \frac{\lambda^{1+2(a-2)}}{\log^{(a-2)}(1/\lambda)}$.

§2) $\tilde{\mathbb{P}}_{(p,q),\lambda}(A_2)$. Assume that A_1 does not occur; then, for a given realization of T , we have

$$P_{T,\lambda}((o,0) \leftrightarrow (B_T(o,1)^c \cap \partial T') \times \mathbb{R}_+ \text{ inside } T') \leq \sum_{y \in \partial T' : d(o,y) \geq 2} (2\lambda)^{d(o,y)}.$$

Also using (6) and (7), we get

$$\begin{aligned} \tilde{\mathbb{P}}_{(p,q),\lambda}(A_2) &\leq \sum_{i=2}^R (2\lambda)^i \cdot \tilde{\mathbb{E}}_{(p,q)}(\#\{y \in \partial T' : d(o,y) = i\} \mid A_1^c) \\ &\leq \sum_{i=2}^R (2\lambda)^i \cdot \tilde{\mathbb{E}}_{(p,q)}(\#\{y \in \partial T' : d(o,y) = i\}) \\ &\leq C \sum_{i=2}^{R-1} \lambda^i \cdot \left(\frac{1}{8\lambda^2}\right)^{(i-1)(3-a)} \cdot (8\lambda^2)^{(a-2)} + C\lambda^R \cdot \left(\frac{1}{8\lambda^2}\right)^{R(3-a)}, \end{aligned}$$

where C is a constant that depends on R but not on λ . This is less than

$$C\lambda^{2(a-2)+1} \sum_{i=2}^{\infty} \lambda^{(i-1)(1-2(3-a))} + C\lambda^{R(1-2(3-a))} \leq C \frac{\lambda^{2(a-2)+1}}{\log^{a-2}(1/\lambda)},$$

by (35).

§3) $\tilde{\mathbb{P}}_{(p,q),\lambda}(A_3)$. By (32),

$$\begin{aligned} \tilde{\mathbb{P}}_{(p,q),\lambda}(A_3) &\leq \tilde{\mathbb{E}}_{(p,q)}(\#T') \cdot e^{-\log^2(1/\lambda)/4} \\ &\leq R \cdot \tilde{\mathbb{E}}_{(p,q)}(\#\{x \in T' : d(o,x) = R\}) \cdot \lambda^{\log(1/\lambda)/4} \\ &\leq R \cdot (1/(8\lambda^2))^{R(3-a)} \cdot \lambda^{\log(1/\lambda)/4} \leq C \frac{\lambda^{2(a-2)+1}}{\log^{a-2}(1/\lambda)} \end{aligned}$$

when λ is small.

§4) $\tilde{\mathbb{P}}_{(p,q),\lambda}(A_4)$. Let $D_1 = \#(B_T(o,1) \cap \partial T')$. As in §2 above,

$$(36) \quad \tilde{\mathbb{P}}_{(p,q),\lambda}(A_4) \leq C \lambda \cdot \tilde{\mathbb{E}}_{(p,q)}(D_1 \cdot I_{\{D_1 \geq 2\}}).$$

Now, conditioned on $\deg(o) = k$, we have

$$\#\{x : d(o,x) = 1, \deg(x) > 1/(8\lambda^2)\} \sim \text{Bin}(k, q[1/(8\lambda^2), \infty)).$$

Let $\theta = C_0 (8\lambda^2)^{a-2} > q[1/(8\lambda^2), \infty)$ as in (6) and let X_i denote a $\text{Bin}(2^i, \theta)$ random variable. The right hand side of (36) is less than

$$(37) \quad C\lambda \sum_{i=0}^{\infty} p[2^i, 2^{i+1}) \cdot \mathbb{E}(X_{i+1} \cdot I_{\{X_{i+1} \geq 2\}}).$$

For a $\text{Bin}(n, p)$ random variable X , we have

$$\begin{aligned}\mathbb{E}(X \cdot I_{\{X \geq 2\}}) &= \sum_{i=2}^n \binom{n}{i} p^i (1-p)^{n-i} \\ &= (np)^2 \sum_{i=2}^n \frac{n-1}{n} \cdot \frac{1}{i(i-1)} \cdot \frac{(n-2)!}{(i-2)!(n-i)!} \cdot p^{i-2} \cdot (1-p)^{n-i} \leq (np)^2.\end{aligned}$$

So, in (37) we use the bounds $p[2^i, 2^{i+1}] \leq (1/2)^{i(a-1)}$ and, for $K \in \mathbb{N}$ to be chosen later,

$$\mathbb{E}(X_{i+1} \cdot I_{\{X_{i+1} \geq 2\}}) \leq \begin{cases} (2^{i+1}\theta)^2 & \text{if } i \leq K; \\ 2^{i+1}\theta & \text{otherwise.} \end{cases}$$

(37) is thus less than

$$\begin{aligned}& C\lambda \sum_{i=0}^K (1/2)^{i(a-1)} \cdot (2^{i+1}\theta)^2 + C\lambda \sum_{i=K+1}^{\infty} (1/2)^{i(a-1)} \cdot 2^{i+1}\theta \\ & \leq C\lambda \sum_{i=0}^K 2^{i(3-a)} \cdot \lambda^{4(a-2)} + C\lambda \sum_{i=K+1}^{\infty} (1/2)^{i(a-2)} \lambda^{2(a-2)} \\ & \leq C\lambda^{1+4(a-2)} \cdot K \cdot 2^K + C\lambda^{1+2(a-2)} \cdot (1/2)^{K(a-2)}.\end{aligned}$$

Now, put $K = \lceil \frac{\log_2(1/\lambda)}{\log 2} \rceil$, where $\log_2 = \log \log$. Since $K \geq \frac{\log_2(1/\lambda)}{\log 2}$, we have

$$C\lambda^{1+2(a-2)} \cdot (1/2)^{K(a-2)} \leq C \frac{\lambda^{1+2(a-2)}}{\log^{a-2}(1/\lambda)}.$$

On the other hand, since $K \leq \frac{2\log_2(1/\lambda)}{\log 2}$, we have

$$C\lambda^{1+4(a-2)} \cdot K \cdot 2^K \leq C\lambda^{1+4(a-2)} \cdot \frac{2\log_2(1/\lambda)}{\log 2} \cdot \log \left(\frac{1}{\lambda} \right) < \frac{\lambda^{1+2(a-2)}}{\log(1/\lambda)}$$

when λ is small.

§5) $\tilde{\mathbb{P}}_{(p,q),\lambda}(A_{51})$. Since arrows from o to x^* arrive as a Poisson process of parameter λ ,

$$\tilde{\mathbb{P}}_{(p,q),\lambda}(A_{51}) \leq \mathbb{P}(\text{Poi}(\lambda \cdot \log^2(1/\lambda)) \geq 3) \leq C(\lambda \log^2(1/\lambda))^3$$

for some $C > 0$ and λ small. Now, when λ is small enough, this is smaller than $\frac{\lambda^{1+2(a-2)}}{\log^{a-2}(1/\lambda)}$ since $1 + 2(a-2) < 3$.

§6) $\tilde{\mathbb{P}}_{(p,q),\lambda}(A_{52})$. This is smaller than

$$\begin{aligned}& \tilde{\mathbb{E}}_{(p,q),\lambda} \left(I \left\{ \begin{array}{l} \deg(o) \leq 1/(8\lambda^2), \\ \#(B_T(o, 1) \cap \partial T') = 1 \end{array} \right\} \cdot P_{T,\lambda} \left(\begin{array}{l} (o, 0) \leftrightarrow \{x^*\} \times \mathbb{R}_+ \\ \text{inside } T' \end{array} \right) \right) \\ & \leq C\lambda \tilde{\mathbb{P}}_{(p,q)}(\exists x \in B_T(o, 1) : \deg(x) > \epsilon(1/\lambda^2) \log(1/\lambda)) \leq C \frac{\lambda^{1+2(a-2)}}{\log^{a-2}(1/\lambda)},\end{aligned}$$

as in §5 of Subsection 3.2.

§7) $\tilde{\mathbb{P}}_{(p,q),\lambda}(A_{53})$. Using Lemma 17, we have

$$\begin{aligned} \tilde{\mathbb{P}}_{(p,q),\lambda}(A_{54}) &\leq \tilde{\mathbb{E}}_{(p,q),\lambda}(I_{A_5} \cdot P_{T,\lambda}((o,0) \leftrightarrow \{x^*\} \times \mathbb{R}_+ \text{ inside } T')) \\ &\quad \cdot \tilde{\mathbb{P}}_{(\delta_{\epsilon(1/\lambda^2)\log(1/\lambda)}, q),\lambda}((o,0) \leftrightarrow \partial T' \text{ inside } T') \\ &\leq C\lambda \cdot \lambda^{\epsilon'} \cdot \tilde{\mathbb{P}}_{(p,q)}(\exists x \in B_T(o,1) : \deg(x) > 1/(8\lambda^2)) \\ &\leq C\lambda^{1+\epsilon'} \cdot \lambda^{2(a-2)} \leq \frac{\lambda^{1+2(a-2)}}{\log^{a-2}(1/\lambda)} \end{aligned}$$

when λ is small.

§8) $\tilde{\mathbb{P}}_{(p,q),\lambda}(A_{54})$. Similarly, this is smaller than

$$\begin{aligned} &\tilde{\mathbb{P}}_{(p,q),\lambda}(A_5 \cap \{\tau_2 < \infty\}) \cdot \tilde{\mathbb{P}}_{(\delta_{\epsilon(1/\lambda^2)\log(1/\lambda)}, q),\lambda}((o,0) \leftrightarrow \partial T' \text{ inside } T') \\ &\leq \tilde{\mathbb{P}}_{(p,q),\lambda}(A_5 \cap \{\tau_1 < \infty\}) \cdot \lambda^{\epsilon'} \leq \frac{\lambda^{1+2(a-2)}}{\log^{a-2}(1/\lambda)}. \end{aligned}$$

5. THE CASE $2 < \alpha \leq 2\frac{1}{2}$, PROOF OF THEOREM 3

In this section the bulk of the density no longer comes from sites which are neighbours of sites with degree of the order (neglecting log terms) $\frac{1}{\lambda^2}$.

5.1. Lower bound. Given Proposition 12 and Proposition 4, our analysis essentially concerns contact processes on random rooted trees, T , produced under law $\tilde{\mathbb{P}}_{(p,q)}$. However first we will consider processes on T with law $\tilde{\mathbb{P}}_q$.

In order to analyze the contact process $(\xi_t^o)_{t \geq 0}$ beginning on such a graph with initially only the root o occupied, we introduce a comparison process $(\eta_t)_{t \geq 0}$, beginning with the same initial conditions. For this process sites become permanently set to value 0 the first time (if ever) that they return to value 0 after having taken value 1. So, in consequence, sites cannot infect sites closer to o than themselves. More precisely, the modified contact process (η_t) is defined as follows. $\eta_t(o) = 1$ until the first death (time $t \in D^o$). And for other sites $y \in T$, y becomes 1 at τ_y if $\tau_y \in D^{z,y}$ and $\eta_{\tau_y}(z) = 1$, where $d(o,y) = d(o,z) + 1$. Furthermore, the value of site y is 1 only on (τ_y, σ_y) , where $\sigma_y := \inf\{t > \tau_y : t \in D^y\}$. Here $D^{z,y}$ and D^y are the Poisson processes generating the original process (ξ_t^o) . It is easy to see that $\eta_t(y) \leq \xi_t^o(y)$ for any time t and any vertex y .

Define

$$X_m := \#\{z : d(o,z) = m, \exists t < \infty \text{ with } \eta_t(z) = 1\}$$

for $m = 0, 1, 2, \dots$. Then $(X_m)_{m \geq 0}$ is a branching process and is in principle easy to analyze.

Throughout this section, we use the notation

$$D = \inf\{t > 0 : \eta_t(o) = 0\}, \quad N = \deg(o).$$

We have the following lemma, which gives a lower bound for the probability $\tilde{\mathbb{P}}_{q,\lambda}(X_1 \geq k)$.

Lemma 20. *For all $k \geq 1$ and $0 < \lambda < \frac{1}{100}$, there exists $C_{6.1}$ universal such that*

$$\tilde{\mathbb{P}}_{q,\lambda}(X_1 \geq k) \geq C_{6.1} \cdot \frac{\lambda^{a-2}}{k^{a-2}}.$$

Proof. Directly calculate to get

$$\begin{aligned} \tilde{\mathbb{P}}_{q,\lambda}(X_1 \geq k) &\geq \tilde{\mathbb{P}}_{q,\lambda}\left(X_1 \geq k, D \geq 1, N \geq \frac{k}{1-e^{-\lambda}}\right) \\ &= e^{-1} \cdot \tilde{\mathbb{P}}_{q,\lambda}\left(N \geq \frac{k}{1-e^{-\lambda}}, X_1 \geq k\right) \\ &\geq e^{-1} \cdot \tilde{\mathbb{P}}_q\left(N \geq \frac{k}{1-e^{-\lambda}}\right) \cdot \tilde{P}\left(\text{Bin}\left(\frac{k}{1-e^{-\lambda}}, 1-e^{-\lambda}\right) \geq k\right) \\ &\geq C \cdot \tilde{\mathbb{P}}_q\left(N \geq \frac{k}{1-e^{-\lambda}}\right) \geq C \cdot \left(\frac{k}{\lambda}\right)^{-a+2}, \end{aligned}$$

as desired. \square

On the other hand, we can get the upper bound for $\tilde{\mathbb{P}}_{q,\lambda}(X_1 \geq k)$. It is a consequence of standard large deviation results.

Lemma 21. *For all $k \geq 1$ and $0 < \lambda < \frac{1}{100}$, there exists $C_{6.2}$ universal such that*

$$\tilde{\mathbb{P}}_{q,\lambda}(X_1 \geq k) \leq C_{6.2} \cdot \frac{\lambda^{a-2}}{k^{a-2}}.$$

Proof. Without loss of generality we suppose that $k = 2^l$ and $\lambda = 2^{-r}$, where l and r are nonnegative integers. Then

$$\begin{aligned} \tilde{\mathbb{P}}_{q,\lambda}(X_1 \geq k) &= \sum_{v=0}^{\infty} \tilde{\mathbb{P}}_{q,\lambda}(X_1 \geq 2^l, N \in [2^{l+v}, 2^{l+v+1})) \\ &\leq \sum_{v=0}^{r-1} \tilde{\mathbb{P}}_{q,\lambda}(X_1 \geq 2^l, N \in [2^{l+v}, 2^{l+v+1})) + \tilde{\mathbb{P}}_{q,\lambda}\left(N \geq \frac{k}{\lambda}\right) \\ &\leq \sum_{v=0}^{r-1} \tilde{\mathbb{P}}_{q,\lambda}(X_1 \geq 2^l, N \in [2^{l+v}, 2^{l+v+1})) + C_1 \cdot \frac{k^{-a+2}}{\lambda^{-a+2}}. \end{aligned}$$

For $v \in \{0, 1, 2, \dots, r-1\}$ fixed, we have

$$\begin{aligned}
& \tilde{\mathbb{P}}_{q,\lambda}(X_1 \geq 2^l, N \in [2^{l+v}, 2^{l+v+1})) \\
& \leq \tilde{\mathbb{P}}_{q,\lambda}(N \geq 2^{l+v}) \cdot \int_0^\infty \mathbb{P}(\text{Bin}(2^{l+v+1}, 1 - e^{-\lambda t}) \geq 2^l) \cdot e^{-t} dt \\
& \leq C_2 \cdot (2^{l+v})^{-a+2} \cdot \int_0^\infty \mathbb{P}(\text{Bin}(2^{l+v+1}, \lambda t) \geq 2^l) \cdot e^{-t} dt \\
& = C_2 \cdot (2^{l+v})^{-a+2} \cdot \left[\int_0^{2^{r-v-2}} \mathbb{P}(\text{Bin}(2^{l+v+1}, \lambda t) \geq 2^l) \cdot e^{-t} dt \right. \\
& \quad \left. + \int_{2^{r-v-2}}^\infty \mathbb{P}(\text{Bin}(2^{l+v+1}, \lambda t) \geq 2^l) \cdot e^{-t} dt \right].
\end{aligned}$$

Note that

$$\int_{2^{r-v-2}}^\infty \mathbb{P}(\text{Bin}(2^{l+v+1}, \lambda t) \geq 2^l) \cdot e^{-t} dt \leq \int_{2^{r-v-2}}^\infty e^{-t} dt = e^{-\frac{2^{r-v}}{4}}.$$

By standard large deviation bounds,

$$\begin{aligned}
\int_0^{2^{r-v-2}} \mathbb{P}(\text{Bin}(2^{l+v+1}, \lambda t) \geq 2^l) \cdot e^{-t} dt & \leq \int_0^{\frac{r-v}{4}} K \cdot e^{-\frac{C_3}{2^{v-r+1}t}} \cdot e^{-t} dt \\
& \leq K \cdot \frac{r-v}{4} \cdot e^{-2 \cdot \frac{C_4}{2 \cdot \frac{v-r+1}{2}}} = K \cdot \frac{r-v}{4} \cdot e^{-C_5 \cdot 2^{\frac{v-r-1}{2}}}
\end{aligned}$$

for universal K, C_3, C_4 and C_5 . So we get

$$\begin{aligned}
\tilde{\mathbb{P}}(X_1 \geq k) & \leq C_1 \cdot \frac{k^{-a+2}}{\lambda^{-a+2}} + \sum_{v=0}^{r-1} C_2 \cdot (2^{l+v})^{-a+2} \cdot e^{-\frac{2^{r-v}}{4}} \\
& \quad + \sum_{v=0}^{r-1} C_2 \cdot (2^{l+v})^{-a+2} \cdot K \cdot \frac{r-v}{4} \cdot e^{-C_5 \cdot 2^{\frac{v-r-1}{2}}} \leq C \cdot \frac{k^{-a+2}}{\lambda^{-a+2}},
\end{aligned}$$

as desired. \square

Having obtained both bounds for the probability $\tilde{\mathbb{P}}_{q,\lambda}(X_1 \geq k)$, we can then get the crucial estimation for $\Phi_\lambda(s)$, the generating function for the branching process (X_n) .

Lemma 22. *For $0 < \lambda < \frac{1}{100}$ fixed, there exist constants $C_{6.3}$ and $s_0 \in (0, 1)$ such that for any $s \in [s_0, 1]$,*

$$\Phi_\lambda(s) \leq 1 - C_{6.3} \lambda^{a-2} (1-s)^{a-2}.$$

Proof. For s close to 1, choose k so that $1 - s \in [2^{-k}, 2^{-k+1})$. Then by Lemma 20 and Lemma 21,

$$\begin{aligned}
\Phi_\lambda(s) &= \sum_{n=0}^{\infty} \tilde{\mathbb{P}}_{q,\lambda}(X_1 = n) \cdot s^n \\
&\leq \sum_{n=0}^{2^k-1} \tilde{\mathbb{P}}_{q,\lambda}(X_1 = n) + \sum_{r=k}^{\infty} \tilde{\mathbb{P}}_{q,\lambda}(X_1 \in [2^r, 2^{r+1})) s^{2^r} \\
&\leq 1 - \tilde{\mathbb{P}}_{q,\lambda}(X_1 \geq 2^k) + \sum_{r=k}^{\infty} \tilde{\mathbb{P}}_{q,\lambda}(X_1 \geq 2^r) \cdot (1 - 2^k)^{2^r} \\
&\leq 1 - C_{6.2} \cdot \frac{(2^k)^{-a+2}}{\lambda^{-a+2}} + \sum_{r=k}^{\infty} C_{6.2} \cdot \frac{(2^r)^{-a+2}}{\lambda^{-a+2}} \cdot e^{-2^{r-k}} \\
&\leq 1 - \frac{C_{6.2}}{2^{a-2}} \cdot \frac{(1-s)^{a-2}}{\lambda^{-a+2}} + \sum_{r=k}^{\infty} e^{-1} \cdot C_{6.2} \cdot \frac{(2^{-a+2})^r}{\lambda^{-a+2}} \\
&= 1 - C_1 \cdot \lambda^{a-2} (1-s)^{a-2} + C_2 \cdot \lambda^{a-2} \cdot \sum_{r=k}^{\infty} (2^{-a+2})^r \\
&\leq 1 - C_1 \cdot \lambda^{a-2} (1-s)^{a-2} + C_3 \cdot \lambda^{a-2} (2^{-a+2})^{k-1} \\
&= 1 - C_3 \cdot \lambda^{a-2} (1-s)^{a-2} + C_3 \cdot \lambda^{a-2} (2^{-k+1})^{a-2} \\
&\leq 1 - C_4 \lambda^{a-2} (1-s)^{a-2},
\end{aligned}$$

as desired. \square

From Lemma 22, we can easily get the following corollary.

Corollary 23. *There exists a constant $C_{6.4}$ such that*

$$\tilde{\mathbb{P}}_{q,\lambda}(X_n \text{ survives}) \geq C_{6.4} \lambda^{\frac{a-2}{3-a}}.$$

Proof. From the standard theory of branching processes (see e.g. [5]), $\beta := \tilde{\mathbb{P}}_{q,\lambda}(X_n \text{ survives})$ satisfies $\Phi_\lambda(1 - \beta) = 1 - \beta$. By Lemma 22, we have $1 - \beta \leq 1 - C_{6.3} \lambda^{a-2} \beta^{a-2}$, so $\beta \geq C \lambda^{\frac{a-2}{3-a}}$, as desired. \square

If we make analagous definitions for process $\{\eta_t : t \geq 0\}$ for T with law $\tilde{\mathbb{P}}_{(p,q)}$, then it is easy to see that

Corollary 24. *For constant $C_{6.5}$ and λ small,*

$$\tilde{\mathbb{P}}_{(p,q)}(X_n \text{ survives}) \geq \frac{C_{6.5}}{2} \lambda^{1+\frac{a-2}{3-a}}.$$

We are now in a position to prove the lower bound of Theorem 3. It follows from Corollary 24 that (if λ is small) with $\tilde{\mathbb{P}}_{(p,q)}$ probability at least $\frac{C_{6.4}}{2} \lambda^{1+\frac{a-2}{3-a}}$ the branching process X survives and so the related process (η_t) will also survive with at least this probability. It is easy to see that

(η_t) cannot survive without infecting a site in T of degree at least $\frac{1}{\lambda^3}$. Given that (ξ_t^o) dominates (η_t) we can conclude that with probability at least $\frac{C_{6.4}}{2} \lambda^{1+\frac{a-2}{3-a}}$, the process (ξ_t^o) infects a site of degree at least $\frac{1}{\lambda^3}$. The lower bound now follows from Proposition 12.

5.2. Upper bound. Let $M = 4C_0$, where C_0 is as in (3)-(7). Given $\lambda > 0$, let $R = R(\lambda) = \lceil 100a \log(1/\lambda) \rceil$.

For the Galton-Watson tree, define $A_1 = \left\{ \deg(o) \geq \frac{(1/\lambda)^{\frac{1}{3-a}}}{M} \right\}$. We have

$$\tilde{\mathbb{P}}_{(p,q)}(A_1) \leq C_0 \cdot M^{a-1} \cdot \lambda^{\frac{a-1}{3-a}} < \lambda^{1+\frac{a-2}{3-a}}$$

when λ is small.

Given a rooted tree T , let

$$T' = \left\{ \begin{array}{l} x \in B_T(o, R) : \text{the geodesic from } o \text{ to } x \text{ does not contain} \\ \text{any site of degree } > (1/\lambda)^{\frac{1}{3-a}}/M \text{ (except possibly } o \text{ and } x) \end{array} \right\}$$

$$\partial T' = \{x \in T' : \deg(x) = 1\}.$$

Thus, $\partial T'$ contains all sites at distance R from o and all sites at distance less than R from o that have degree larger than $(1/\lambda)^{\frac{1}{3-a}}/M$. If $\deg_T(o) \leq 1/(8\lambda^2)$, Lemma 7 implies that

$$P_{T,\lambda}((o, 0) \leftrightarrow \partial T' \times \mathbb{R}_+) \leq \sum_{i=1}^{\infty} (2\lambda)^i \cdot \#\{x \in \partial T' : d(o, x) = i\}.$$

Using this, we get

$$\begin{aligned} & \tilde{\mathbb{P}}_{(p,q),\lambda}((o, 0) \leftrightarrow \partial T' \times \mathbb{R}_+ \mid A_1^c) \\ & \leq \sum_{i=1}^R (2\lambda)^i \cdot \tilde{\mathbb{E}}_{(p,q),\lambda}(\#\{x \in \partial T' : d(o, x) = i\} \mid A_1^c) \\ & \leq \sum_{i=1}^R (2\lambda)^i \cdot \tilde{\mathbb{E}}_{(p,q),\lambda}(\#\{x \in \partial T' : d(o, x) = i\}) \\ & \leq \left[\sum_{i=1}^{R-1} (2\lambda)^i \cdot \left(C_0 \left(\frac{(1/\lambda)^{\frac{1}{3-a}}}{M} \right)^{3-a} \right)^{i-1} \cdot C_0 \left(\frac{(1/\lambda)^{\frac{1}{3-a}}}{M} \right)^{-(a-2)} \right] \\ & \quad + (2\lambda)^R \cdot \left(C_0 \left(\frac{(1/\lambda)^{\frac{1}{3-a}}}{M} \right)^{3-a} \right)^R \\ & \leq C\lambda^{\frac{a-2}{3-a}+1} \sum_{i=1}^R \frac{2^i C_o^{i-1}}{M^{i-1}} + \frac{(2C_0)^R}{M^{(3-a)R}} \\ & \leq C\lambda^{\frac{a-2}{3-a}+1} \sum_{i=1}^R \frac{2^i C_o^{i-1}}{(4C_0)^{i-1}} + 2^{-R} \leq C\lambda^{\frac{a-2}{3-a}+1} + \lambda^{100a \log(2)} \leq C\lambda^{\frac{a-2}{3-a}+1}. \end{aligned}$$

Since $\tilde{\mathbb{P}}_{(p,q),\lambda}(\xi_t^o \neq \emptyset \ \forall t) \leq \tilde{\mathbb{P}}_{(p,q)}(A_1) + \tilde{\mathbb{P}}_{(p,q),\lambda}((o, 0) \leftrightarrow \partial T' \times \mathbb{R}_+ \mid A_1^c)$, the upper bound is now established.

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